OSCILLATIONS

SOME SYSTEMS EXECUTING SIMPLE HARMONIC MOTION

PHYSICAL PENDULUM

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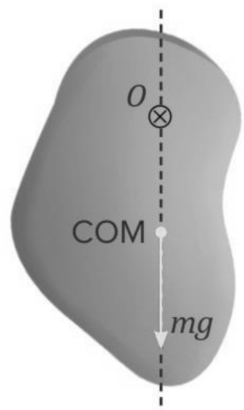
When a solid object is hung from a fixed point and swings back and forth around the axis that goes through that hanging point, it is referred to as a physical pendulum or a compound pendulum.

When the object is in a balanced state, its center of mass is directly below the point it's hanging from. This means there is no twisting force caused by gravity around the hanging point.

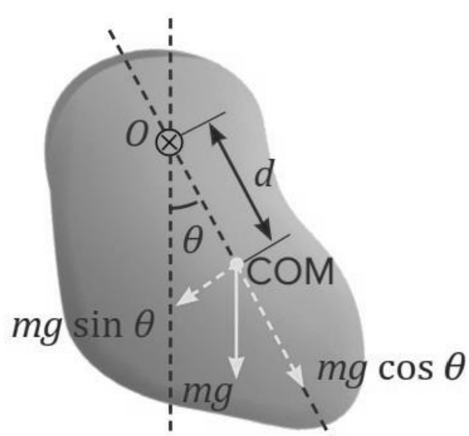


Equilibrium position

Imagine the object is moved a little bit from its central position, and this shift is represented in the second figure.

The torque acting on the body about point O is given by,

$\tau_0 = -mg\sin\theta \times d$



Where, is the distance between the point of suspension and the Centre of mass of the body.

$I_{hinge}\alpha = -mgd\sin\theta$

$\alpha = -\frac{mgd}{I_{Hinge}}\sin\theta$

Since $\sin\theta \approx \theta$ for small angles,

$\alpha = -\omega^2\theta$

Where, $\omega = \sqrt{\frac{mgd}{I_{\text{Hinge}}}}$

Hence, the time period of the SHM of the rigid body is given by,

$$T = 2\pi\sqrt{\frac{I_{\text{Hinge}}}{mgd}}$$

If we already know the moment of inertia of the object around its center (I_c), we can use the parallel axis theorem to express the following:

$$T = 2\pi\sqrt{\frac{I_C + md^2}{mgd}}$$

Question.

In the illustration provided, a circular disk with mass denoted as M and a radius represented as R is affixed at a pivotal point O. When it undergoes a slight rotational displacement around the point O and subsequently released, determine the time period for the ensuing oscillations.

Solution.

The distance of the Centre of mass from the point of suspension, $d = \frac{R}{2}$

The moment of inertia of the disc about the Centre of mass, $I_C = \frac{MR^2}{2}$

Hence, in accordance with the principle known as the parallel axis theorem, the moment of inertia pertaining to the circular disc concerning the point O can be expressed as follows:

$$I_O = I_C + Md^2$$

$$I_O = \frac{MR^2}{2} + M\left(\frac{R}{2}\right)^2$$

$$I_O = \frac{3MR^2}{4}$$

Hence, the time period of the oscillations is given by,

$$T = 2\pi\sqrt{\frac{I_O}{Mgd}}$$

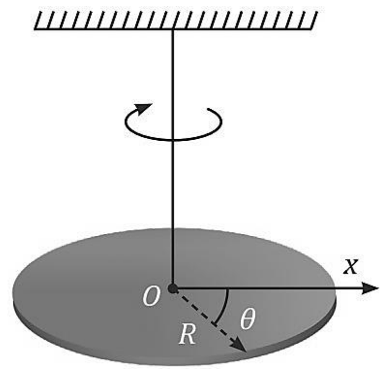
$$T = 2\pi\sqrt{\frac{\frac{3MR^2}{4}}{Mg\left(\frac{R}{2}\right)}}$$

$$T = 2\pi\sqrt{\frac{3R}{2g}}$$

Torsional Pendulum

Contemplate an elongated object suspended through a weightless cord, as illustrated. When this object is gently rotated about the cord, which serves as the axis of rotation, it becomes apparent that the restorative torque induced by the pendulum's cord is directly proportionate to the angle of rotation (θ).

Thus,



$$\tau \propto -\theta$$
$$\tau = -C\theta$$

Where, C is known as the torsional constant.
If the angular acceleration of the body at the time of release is α, then,

$$I\alpha = -C\theta$$
$$\Rightarrow \alpha = -\omega^2\theta$$

When, $\omega = \sqrt{\frac{C}{I}}$

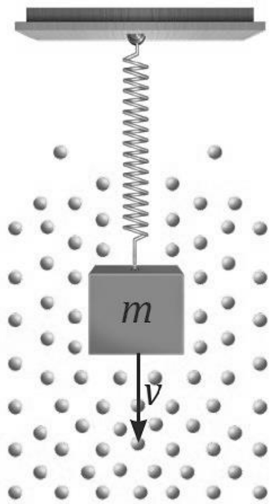
Hence, the time period of the torsional SHM is given by, $T = 2\pi\sqrt{\frac{I}{C}}$.

At a specific point in time, if the cord under examination is wound or twisted by an angle θ, then the potential energy that becomes stored within it can be mathematically expressed as: $U = \frac{1}{2}C\theta^2$

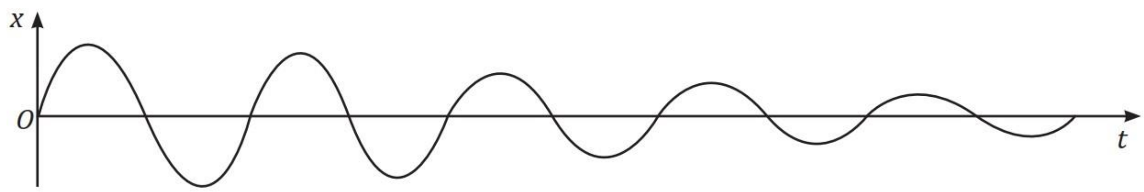
Damped Oscillations

Damping is when something outside, like friction or drag, tries to slow down an object's back-and-forth movement. Simple Harmonic Motion (SHM) is a simplified way of looking at how things move in a repetitive pattern, but in reality, there are often other forces involved in making things slow down or stop.

In a setup where a block is attached to a spring and moves up and down, the air molecules around it constantly collide with the block, making it lose some of its energy. This causes the block's bouncing motion to gradually become weaker and weaker, with smaller swings each time.

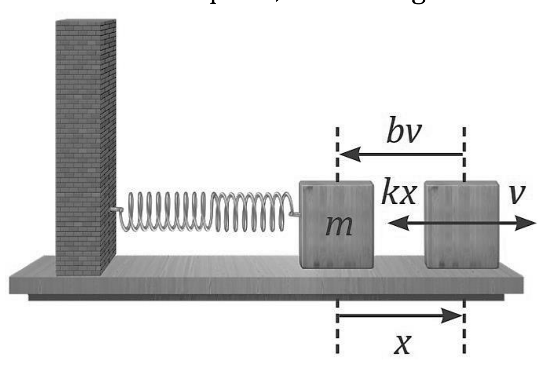


A typical displacement vs time plot of damped oscillations is shown in the figure below.



Think about a block moving back and forth on a flat surface because of a spring's push, like in the picture on the next page. There's also a force that slows the block down, and this force depends on how fast the block is moving. It always pushes in the opposite direction of the block's motion.

At a certain time (t), if the block is at a position x away from its middle point, it's moving at a certain speed, as you can see in the figure.



Thus, the net force acting on the block is given by,

$$F = -kx - bv = ma$$

Where, b is known as the damping constant.

The differential equation of the motion of the block is given by,

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

The amplitude of the damped oscillations at time t is given by,

$$A(t) = A_0 e^{-\frac{bt}{2m}}$$

Thus, the energy of the oscillations at time t is given by,

$$E = \frac{1}{2} k A_0^2 e^{-\frac{bt}{m}}$$

The angular frequency of the oscillations is also decreased. It is given by, $\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$.

Superposition Principle

The superposition principle simply means that when you have multiple waves, you can find the total displacement of a particle by adding up the displacements caused by each wave, like adding vectors.

For vector response

Suppose there are two displacements, $\vec{y}_1$ and$\vec{y}_2$, which occurred due to two different reasons. The net displacement would be, $\vec{y}_{net} = \vec{y}_1 + \vec{y}_2$

For scalar response

According to the principle of superposition, the net response (r_{net})due to all the other individual responses ($r_1, r_2, r_3, \dots \dots, r_n$) will be,

$$r_{net} = r_1 + r_2 + r_3 + \cdots + r_n$$

Superposition of two SHMs

Same direction (same straight line) and same frequency:

Let us contemplate two Simple Harmonic Motions (SHMs), both having an identical frequency and direction, originating from two separate harmonic oscillators. They are mathematically represented as $x_1 = A_1 \sin(\omega t)$ and $x_2 = A_2 \sin(\omega t + \phi)$, wherein ϕ signifies the phase disparity between these two SHMs. When these two SHMs are amalgamated or combined, the overall displacement is determined as follows:

$$x = x_1 + x_2$$

$$x = A_1 \sin \omega t + A_2 \sin(\omega t + \phi)$$

$$x = A_1 \sin \omega t + A_2 [\sin \omega t \cos \phi + \cos \omega t \sin \phi]$$

$$x = [A_1 + A_2 \cos \phi] \sin \omega t + [A_2 \sin \phi] \cos \omega t$$

By denoting $[A_1 + A_2 \cos \phi] = A \cos \alpha$ and $A_2 \sin \phi = A \sin \alpha$, we get,

$$x = A\cos\alpha \sin\omega t + A\sin\alpha\cos\omega t$$
$$x = A[\sin\omega t\cos\alpha + \cos\omega t\sin\alpha]$$
$$x = A\sin(\omega t + \alpha)$$

Therefore, when two SHMs with matching frequency and direction are combined, they yield yet another SHM of the same frequency.

Earlier, we assumed the following:

$$\left[A_1 + A_2\cos\phi\right] = A\cos\alpha \quad .....(i)$$

$$A_2\sin\phi = A\sin\alpha \quad .....(ii)$$

By squaring and adding these two equations, we get,

$$A^2 = \left[A_1 + A_2\cos\phi\right]^2 + \left[A_2\sin\phi\right]^2$$
$$A = \sqrt{\left[A_1 + A_2\cos\phi\right]^2 + \left[A_2\sin\phi\right]^2}$$
$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$$

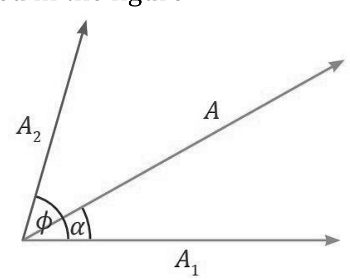
On dividing equation by equation, we get

$$\tan\alpha = \frac{A_2\sin\phi}{A_1 + A_2\cos\phi}$$
$$\alpha = \tan^{-1}\left(\frac{A_2\sin\phi}{A_1 + A_2\cos\phi}\right)$$

Geometrical method (Phasor diagram):

Let us examine two Simple Harmonic Motions, described by the equations $x_1 = A_1\sin(\omega t)$ and $x_2 = A_2\sin(\omega t + \phi)$, where ϕ represents the phase difference between these two motions. When these two SHMs are superimposed, the resultant response can be represented as $x = A\sin(\omega t + \alpha)$. To determine the unknown values A and α, let us consider two phasors, A_1 and A_2, forming an angle ϕ between them. These phasors can be combined as vectors, as illustrated in the figure.

Now, by applying the formula to find the resultant vector, we get,



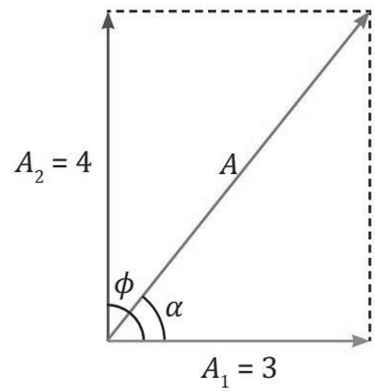
$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$$

And, $\alpha = \tan^{-1}\left(\frac{A_2\sin\phi}{A_1 + A_2\cos\phi}\right)$

Question.

A particle experiences the influence of two SHMs, characterized by their respective displacements: $x_1 = 3\sin(\omega t + 30°)$ and $x_2 = 4\sin(\omega t + 120°)$. Our objective is to determine the equation that describes the displacement of the resultant SHM.

Solution.



The net displacement is given by,

$$x_{net} = 3\sin\left(\omega t + 30^\circ\right) + 4\sin\left(\omega t + 120^\circ\right)$$

The phase difference between x_2 and x_1 is given by,

$$\omega t + 120^\circ - \omega t - 30^\circ = 90^\circ$$

The phasor diagram of the given SHMs is as shown in the figure.

The amplitude of the resultant SHM is given by

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi} = \sqrt{3^2 + 4^2 + 2\times 3\times \cos 90^\circ}$$

$$A = 5$$

Also,

$$\tan\alpha = \frac{4}{3}$$

$$\alpha = 53^\circ$$

The angle made by x_{net} with the x_1 is 53°. Further, the phase angle of x_1 is 30°.

Hence, the displacement of the resultant SHM is given by,

$$x_{net} = 5\sin\left(\omega t + 30^\circ + 53^\circ\right)$$

$$x_{net} = 5\sin\left(\omega t + 83^\circ\right)$$