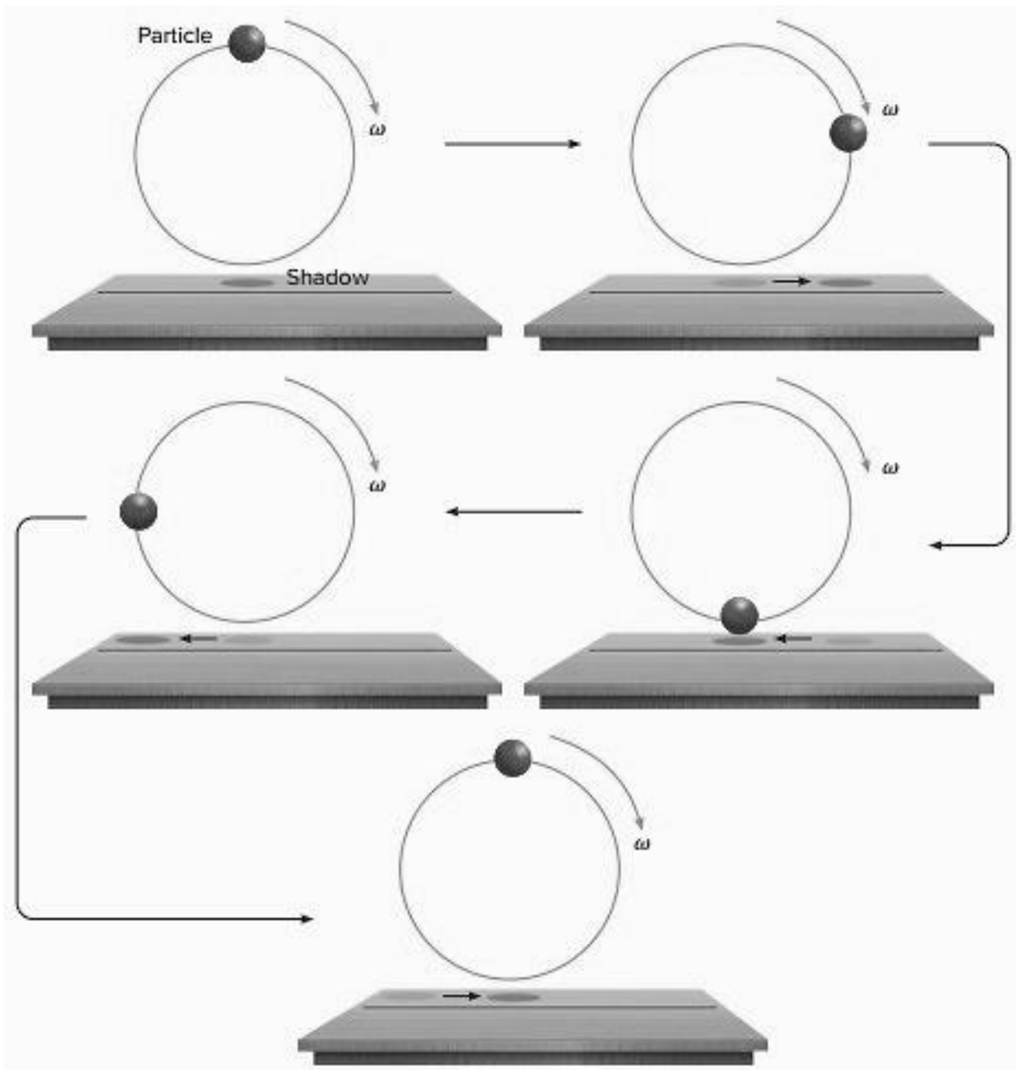
OSCILLATIONS

SIMPLE HARMONIC MOTION AND UNIFORM CIRCULAR MOTION

SHM AS A PROJECTION OF UNIFORM CIRCULAR MOTION

Think about a particle moving around in a circle at a steady speed, like a clock's minute hand going clockwise. The shadow of this particle on a flat surface moves back and forth. If we look at different moments while the particle is moving in a circle, we notice the following:



As the particle goes around the circle once, its shadow moves to the right, then back to where it started, and then to the left before returning to the starting point. The shadow of the circular motion follows a back-and-forth pattern, like it's swinging within a certain distance. This swinging motion has a size (amplitude) that's the same as the circle's radius.

It's important to understand that both the particle's motion and its shadow's motion repeat over and over, which is what we call 'periodic.' However, the circular motion itself isn't the same as swinging back and forth, but its shadow does.

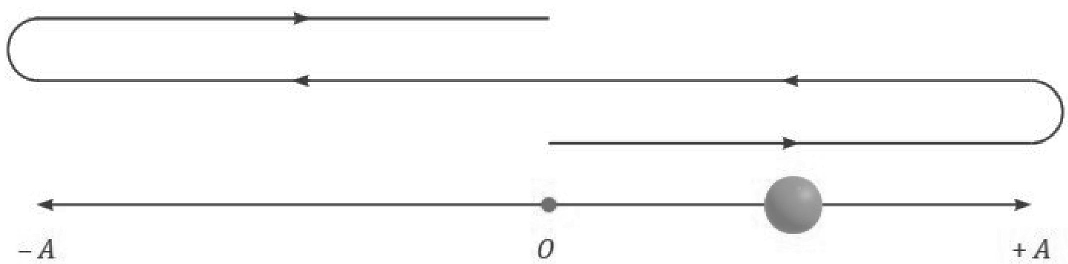
Some Important Terminologies in SHM

Amplitude (A)

The farthest the particle moves away from its starting point is called the 'amplitude' of the back-and-forth motion. In another way, you can think of this amplitude as the same as the circle's radius in the circular motion that matches the back-and-forth motion of the particle.

Time period (T)

The smallest time interval after which the motion of the particle that is executing SHM gets repeated is known as the time period of the SHM.



If the angular velocity of the particle exhibiting UCM is ω, then the time period of the corresponding SHM is, $T = \frac{2\pi}{\omega}$

Frequency (f)

The quantity that represents the count of complete cycles made by an oscillating body in a given amount of time is referred to as the frequency of the SHM.

$f = \frac{1}{T}$

Its SI unit is Hz or s^{-1}.

Angular frequency (ω)

The angular frequency (ω) in a SHM corresponds to the angular velocity in the UCM connected to it. We will explore further that it represents how fast the phase angle of the SHM is changing. Its SI unit is rads^{-1}.

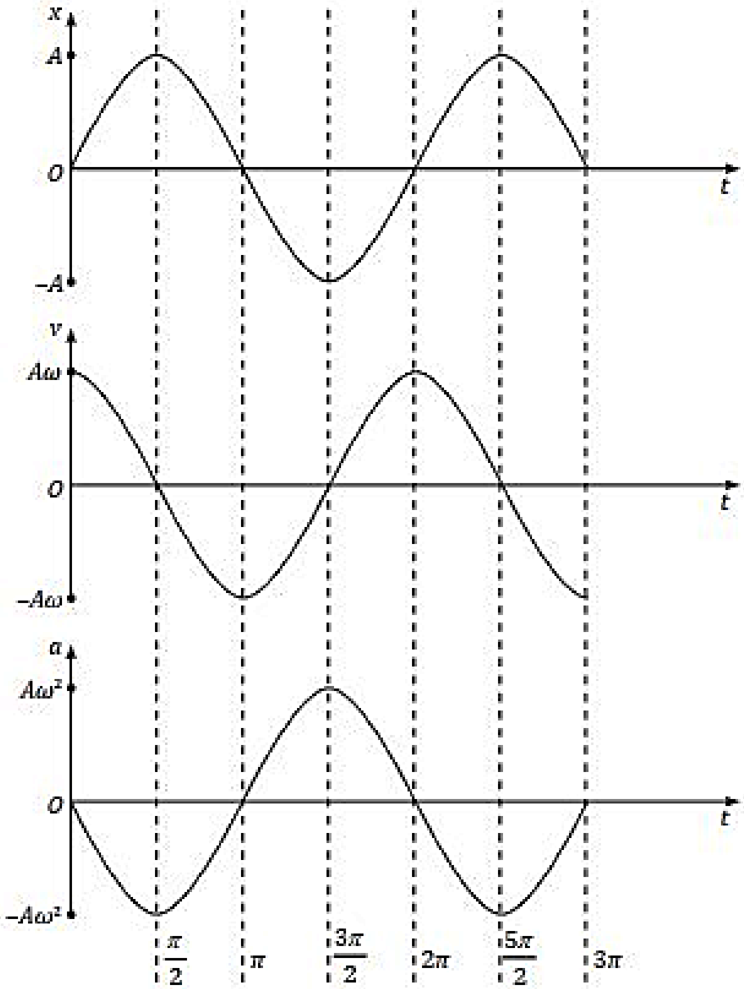
Graphical Representation

If the position of the particle is described as x = A sin ωt, then the formulas for the speed and how fast it's changing speed are like this:

$v = \frac{dx}{dt} = A\omega \cos \omega t = A\omega \sin\left(\omega t + \frac{\pi}{2}\right)$

$a = \frac{dv}{dt} = -A\omega^2 \sin \omega t = A\omega^2 \sin(\omega t + \pi)$

The graphs of these quantities with respect to time are as follows:

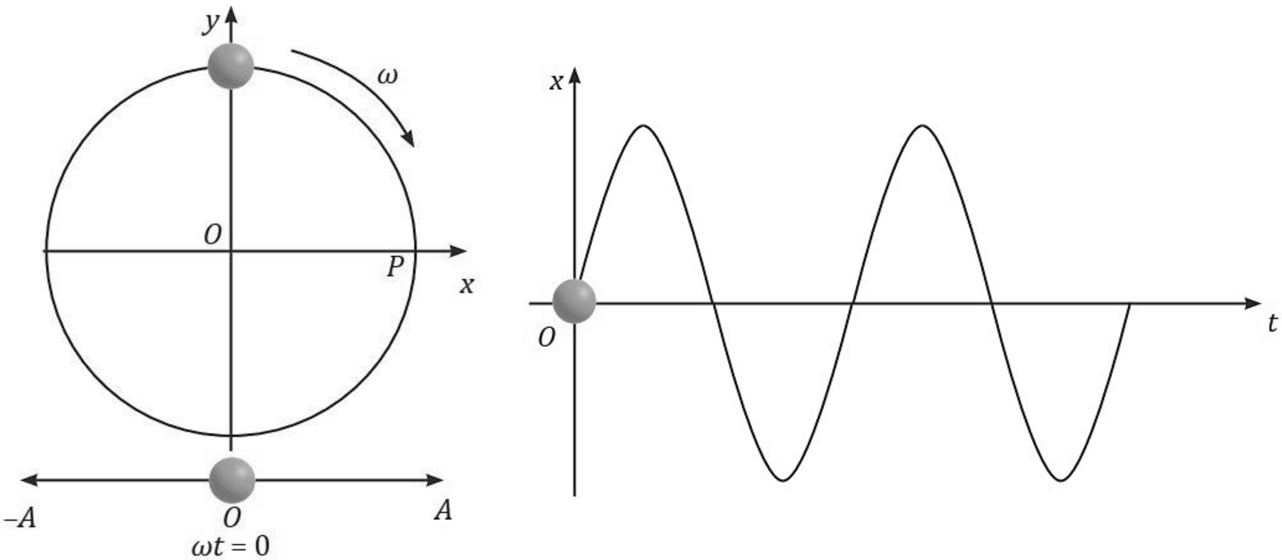


When we look at the timing of the displacement, velocity, and acceleration, we can see some relationships. The velocity is a little bit ahead of the displacement by a quarter rotation ($\frac{\pi}{2}$), the acceleration is a bit ahead of the velocity by a quarter rotation, and the velocity is a bit behind the acceleration by a quarter rotation. Additionally, the acceleration is a little ahead of the displacement by a full rotation (π).

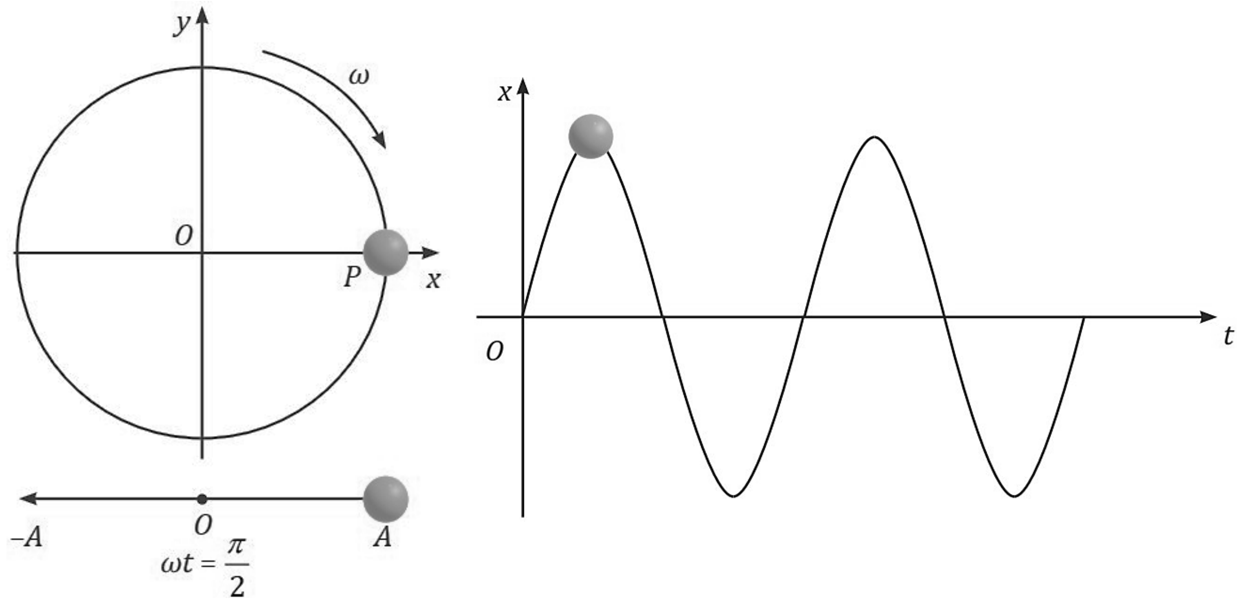
Characteristics of SHM

Let's analyze a particle undergoing SHM, as illustrated in the diagram. The particle's position is described by the equation $x = A \sin(\omega t + \phi)$. At the beginning of its journey, the particle initiates Uniform Circular Motion (UCM) from the North Pole, where $\phi = 0$. Consequently, the expression for its displacement becomes:

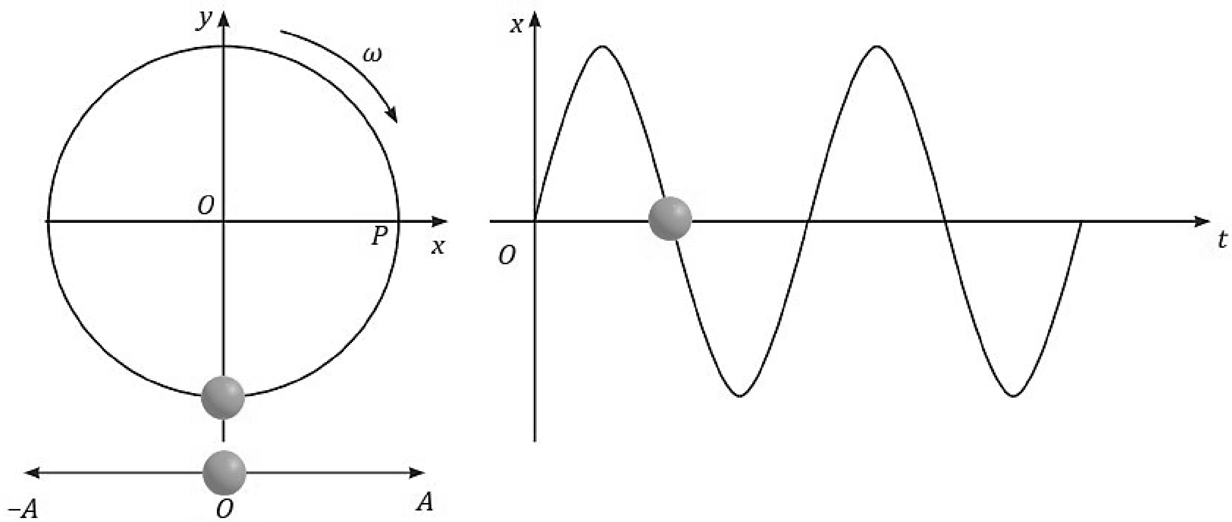
$x = A \sin \omega t$



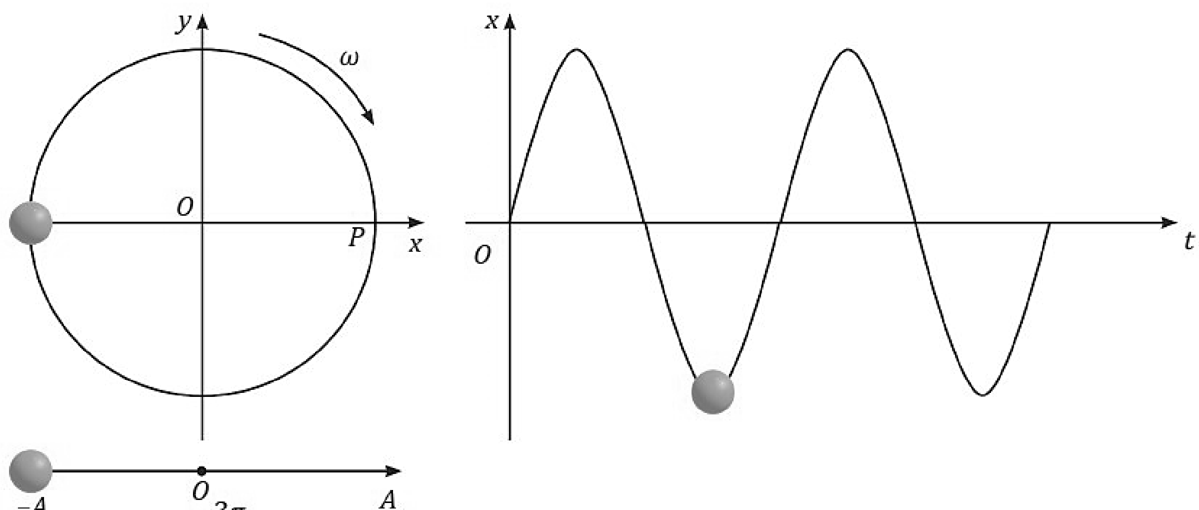
As the particle progresses from its starting point on the Uniform Circular Motion (UCM) to reach point P in its linked SHM, it travels from the central position O to the farthest point A. In the time-dependent representation of the particle's motion, this journey is visualized as the particle moving from position O to the highest point of the sine curve, as illustrated in the diagram.



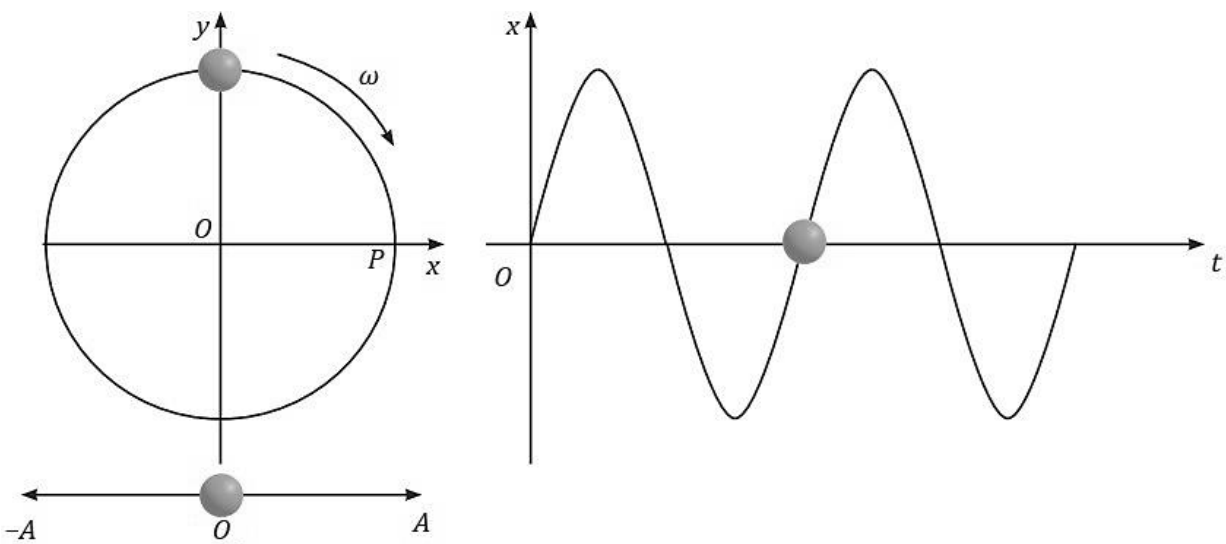
Likewise, the particle's subsequent movements in the SHM from A to O, O to -A, and -A back to O, along with the corresponding UCM and the time-dependent graph, are displayed.



$\omega t = \pi$



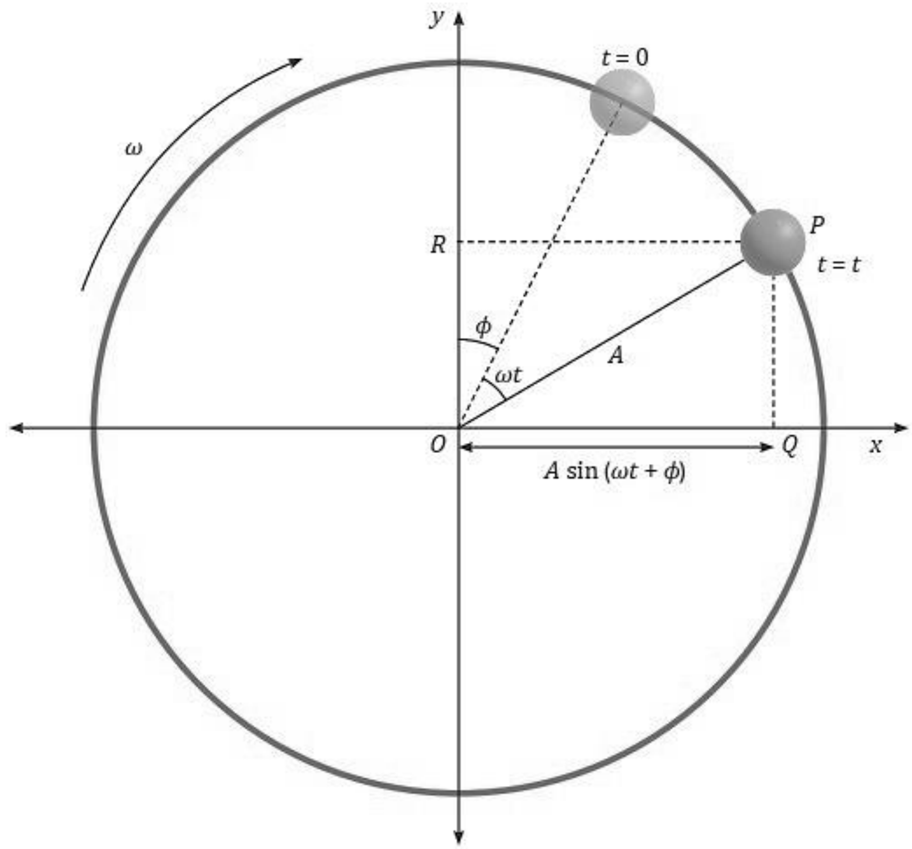
$\omega t = \frac{3\pi}{2}$



$\omega t = 2\pi$

SHM as a Projection of UCM

Let's say there's a particle going around in a circle with the center at the point (0, 0) and a distance of A from the center. At first, it's located at a spot that's at an angle ϕ from the vertical y-axis, and it's moving in a clockwise direction. The particle has an angular speed of ω, and in the time period of t, it covers an angular distance of ωt.



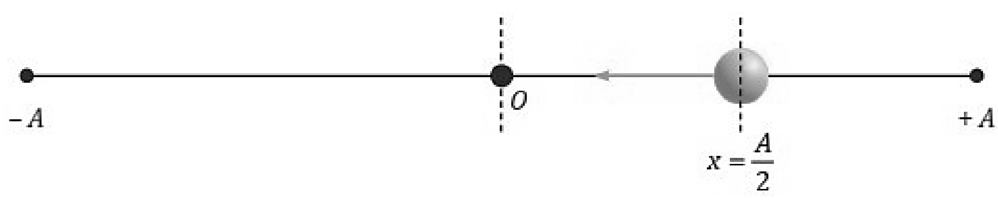
The projection of the particle on the x-axis is given by,
$x = A \sin(\omega t + \phi)$

It's like when the particle moves in a circle, the way it swings back and forth along a line is just like simple back-and-forth motion with a certain size (amplitude A) and speed (angular frequency ω).

So, even though it's circling, its shadow along a line goes back and forth like regular swinging.

Question.

A particle initially at $x = \frac{A}{2}$ executes SHM with angular frequency ω in the direction as shown, where A is the amplitude. Find out the displacement-time equation of the particle.



Solution.

Suppose the particle's back-and-forth motion can be described by the equation x = A sin (ωt + ϕ). We know that the particle starts its motion when it's at the position $x = \frac{A}{2}$.

By putting this condition into the equation, we get,

$A = \sin \emptyset = \frac{A}{2}$

$\sin \emptyset = \frac{1}{2}$

$\emptyset = \frac{\pi}{6}\ or\ \frac{5\pi}{6}$

Because the particle is going in the direction of the left on the x-axis, its velocity is in the negative direction.

The general expression of velocity is given by,

$v = A\omega \cos (\omega t + \phi)$

At $t = 0$,

$v = A\omega \cos \phi$

Among the obtained values of ϕ, the value of cos ϕ would be negative only for $\phi = \frac{5\pi}{6}$
Hence, the displacement-time equation of the given SHM is as follows:
$x = A \sin\left(\omega t + \frac{5\pi}{6}\right)$