LAW OF RADIOACTIVE DECAY

- Radioactivity is a random process
- A particular nucleus can decay at anytime between $t = 0$ to $t \to \infty$
- It is impossible to predict when a particular nucleus will decay/disintegrate
- Rutherford & Soddy conducted experimental study on different kinds of nuclei & gave a statistical law:

  Rate of decay/disintegration $\propto$ No. of active nuclei in the sample
- The rate of decay/disintegration means the number of decay per unit time

- Let N_0 be the no. of active nuclei at time $t = 0$ and N be the no. of active nuclei at any time t. Since "Rate of decay/disintegration $\propto$ No. of active nuclei in sample", we can write:

$-\frac{dN}{dt} \propto N$ [The negative sign shows that upon disintegration, no. of active nuclei decreases]

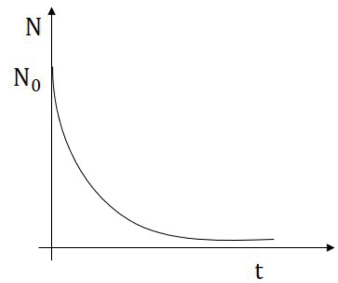
$\Rightarrow -\frac{dN}{dt} = \lambda N$[Where $\lambda =$ Decay constant]

$\Rightarrow \int_{N_0}^{N} \frac{dN}{N} = -\lambda \int_0^t dt$

$\Rightarrow \ln \frac{N}{N_0} = -\lambda t$

$\Rightarrow \frac{N}{N_0} = e^{-\lambda t} \Rightarrow N = N_0 e^{-\lambda t}$

$N = N_0 e^{-\lambda t}$



$N \to$ No. of nuclei left undecayed at time t

$N_0 \to$ Initial No. of undecayed nuclei ($t = 0$)

At $t = \infty \to N = 0$ Complete Decay

Rate of disintegration,

$$R = -\frac{dN}{dt} = \lambda N$$

$$\Rightarrow \lambda N = \lambda N_0 e^{-\lambda t}[\text{ Since } N = N_0 e^{-\lambda t}, \frac{dN}{dt} = -\lambda N_0 e^{-\lambda t}]$$

$$\Rightarrow R = R_0 e^{-\lambda t}$$

Where,

$$R_0 = \text{ Rate of disintegration at time } t = 0$$

$$R = R_0 e^{-\lambda t}$$

The half life of radioactive decay is the time in which half of active nuclei decays. It means at

$$t = t_{1/2} \to N = \frac{N_0}{2}$$

We know,

$$N = N_0 e^{-\lambda t}$$

Putting $N = \frac{N_0}{2}$ for $t = t_{1/2}$ in the above equation, we get,

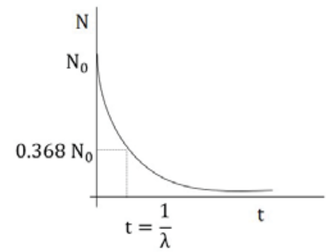
$$\frac{N_0}{2} = N_0 e^{-\lambda t_{1/2}}$$

$$\ln 2 = \lambda t_{1/2}$$

$$t_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda} [\because \ln 2 = 0.693]$$

$$t_{1/2} = \frac{\ln 2}{\lambda}$$

Half Life of Radioactive Decay



Number of nuclei left after 'n' half lives:

After 1 half life,

$$N = \frac{N_0}{2} = \frac{N_0}{2^1}$$

After 2 half life,

$$N = \frac{N_0}{4} = \frac{N_0}{2^2}$$

After 3 half life,

$$N = \frac{N_0}{8} = \frac{N_0}{2^3}$$

After 'n' half life,

$$N = \frac{N_0}{2^n}$$

No. of half lives,

$$n = \frac{t}{t_{1/2}}$$

For e.g. : Let $t = 16$ days

$$t_{1/2} = 4 \text{ days}$$

$$\therefore n = \frac{t}{t_{1/2}} = \frac{16}{4} = 4$$

Unit of Activity

Becquerel = $1\ Bq = 1$ disintegrations/s

Rutherford = 10^6 disintegrations/s = $10^6\ Bq$

Curie = $1\ Ci = 3 \times 10^{10}$ disintegrations/s = $3 \times 10^{10} Bq$

$$R = R_0 e^{-\lambda t}$$

Mean Life of Radioactive Decay

The mean life of radioactive decay of an element is defined as,

$$\tau = \frac{\text{Sum of lives of all nuclei}}{N_0} = \frac{S}{N_0}$$

Let at time t, number of active nuclei be N and in time dt, dN nuclei disintegrates.

We know,

$$\frac{-dN}{dt} = \lambda N$$

$$dN = \lambda N dt$$

[Here, we have considered the magnitude only]

Total life span of dN nuclei = $(dN)t = (\lambda N dt)t$

$$= (\lambda N t)dt$$

As $N = N_0 e^{-\lambda t}$, sum of lives of all nuclei becomes.

$$S = \int_0^\infty \lambda N t dt$$

$$S = \int_0^\infty \lambda N_0 e^{-\lambda t} t dt = \lambda N_0 \int_0^\infty t e^{-\lambda t} dt$$

Integrating by parts, we get,

$$S = \lambda N_0 [t \int e^{-\lambda t} dt - \int (1 \times \int e^{-\lambda t} dt) dt]$$

$$S = \lambda N_0 \int t \int e^{-\lambda t} dt - \int (\int e^{-\lambda t} dt) dt$$

$$= \lambda N_0 [t \left(\frac{e^{-\lambda t}}{-\lambda}\right) - \int \frac{e^{-\lambda t}}{-\lambda} dt]$$

$$= \lambda N_0 [t \cdot \frac{e^{-\lambda t}}{-\lambda} - (\frac{-1}{\lambda} (\frac{e^{-\lambda t}}{-\lambda}))]_0^\infty$$

$$= \lambda N_0 [\frac{t e^{-\lambda t}}{-\lambda} - \frac{e^{-\lambda t}}{\lambda^2}]_0^\infty$$

$$= -N_0[te^{-\lambda t} + \frac{e^{-\lambda t}}{\lambda}]_0^\infty$$
$$= -N_0([te^{-\lambda t}]_0^\infty + [\frac{e^{-\lambda t}}{t}]_0^\infty)$$
$$= N_0([\frac{t}{e^{\lambda t}}]_0^\infty + \frac{1}{\lambda}[\frac{1}{e^{\lambda t}}]_0^\infty$$
$$= -N_0([0] + \frac{1}{2}[0 - 1])$$
$$= \frac{N_0}{\lambda}$$

Mean life of radioactive decay,

$$\tau = \frac{s}{N_0} = \frac{1}{\lambda}$$

$$\tau = \frac{1}{\lambda}$$

Half life,

$$t_{1/2} = \frac{\ln 2}{\lambda}$$

Mean life,

$$\tau = \frac{1}{\lambda}$$
$$\therefore t_{1/2} = \frac{\ln 2}{\lambda} = \tau \ln 2$$

Since $\ln 2 < 1$, from the relation given above, we can say that $\tau > t_{1/2}$.
Thus, "Mean Life > Half life".

Questions based on law of radioactive decay

Ex. A radioactive element decays by $\beta -$ emission. A detector records n beta particles in 2 s and in next 2 s it records $0.75n$ beta particles. Find mean life correct to nearest whole number. Given ln $2 = 0.6931$, $ln\, 3 = 1.0986$.

Sol. Let at $t = 0$, number of active nuclei $= N_0$
Now, one β particle is emitted for each disintegration.
It is given that a detector records n beta particles in 2s.
Therefore, at $t_0 = 2s$, number of nuclei remaining, $N = N_0 e^{-\lambda t_0}$
Number of nuclei disintegrated, $N_d = N_0 - N_0 e^{-\lambda t_0}$
Hence, number of β particle emitted, $n = N_d = N_0(1 - e^{-\lambda t_0})$ .....(1)
At $t = 2t_0 = 4s$,
Number of nuclei remaining, $N = N_0 e^{-\lambda(2t_0)}$
Number of nuclei disintegrated b/w $t = 2s$ and $t = 4s$, $N'_d = N_0 e^{-\lambda t_0}(1 - e^{-\lambda t_0})$
It is also given that the detector records 0.75n beta particles in next 2s
i.e., b/w $t = 2$ s and $t = 4s$.
$\therefore 0.75n = N_0 e^{-\lambda t_0}(1 - e^{-\lambda t_0})$ ......(2)
Dividing equation (1) by equation (2), we get,

$$\frac{1}{0.75} = \frac{1}{e^{-\lambda t_0}}$$
$$e^{\lambda t_0} = \frac{4}{3}$$
$$\lambda t_0 = \ln 4/3$$

Mean life,

$$\tau = \frac{1}{\lambda} = \frac{t_0}{\ln\left(\frac{4}{3}\right)} = \frac{2}{2\ln 2 - \ln 3} = \frac{2}{2 \times 0.6931 - \ .0986} = 7\text{ s}$$
$$\tau = 7s$$

Ex. I^{131} is an isotope of Iodine that β decays to an isotope of Xenon with a half-life of 8 days. A small amount of a serum labelled with I^{131} is injected into the blood of a person. The activity of the amount of I^{131} injected was 2.4×10^5 *Becquerel* (Bq). It is known that the injected serum will get distributed uniformly in the blood stream in less than half an hour. After 11.5 hours, 2.5 ml of blood is drawn from the person's body, and gives an activity of 115 Bq. The total volume of blood in the person's body, in liters is approximately (you may use $e^x \approx 1 + x$ for $x \ll 1$ and $\ln 2 \approx 0.7$)

Sol. Initial activity of I^{131}, $R_0 = 2.4 \times 10^5$ Bq

Half life of I^{131}, $t_{1/2} = 8$ days

The injected serum i.e., I^{131} will get distributed uniformly in the blood stream in $t = 12$hrs

Activity of I^{131} in entire blood after time t, $R = R_0 e^{-\lambda t}$

$$R = R_0 e^{-\lambda t} = 2.4 \times 10^5 [e^{-\frac{0.7\times 12}{8\times 24}}] \qquad \because \lambda = \frac{\ln 2}{t_{1/2}} \approx \frac{0.7}{8 \times 24}$$

$$= 2.4 \times 10^5 (e^{-\frac{0.7}{16}})$$

$$= 2.4 \times 10^5 \left(1 - \frac{0.7}{16}\right) \qquad \because e^x \approx 1 + x$$

$$= 2.4 \times 10^5 \times \frac{15.3}{16}$$

Let total volume of blood in the body be V litres

If R is the activity in V litres of blood, then activity in 2.5ml of blood is, $\frac{R}{V} \times 2.5 \times 10^{-3} = 115$Bq

$\Rightarrow V = 5$ litres