GAUSS' LAW

GAUSS'S LAW IN ELECTROSTATICS OR GAUSS'S THEOREM

This law was stated by a mathematician Karl F Gauss. This law gives the relation between the electric field at a point on a closed surface and the net charge enclosed by that surface This surface is called Gaussian surface. It is a closed hypothetical surface. Its validity is shown by experiments. It is used to determine the electric field due to some symmetric charge distributions

Statement and Details:

Gauss's law is stated as given below. The surface integral of the electric field intensity over any closed hypothetical surface (called Gaussian surface) in free space is equal to $\frac{1}{\varepsilon_0}$ times the total charge enclosed within the surface. Here, ε_0 is the permittivity of free space.

If S is the Gaussian surface and $\sum_{i=1}^{n} q_i$ is the total charge enclosed by the Gaussian surface, then according to Gauss's law,

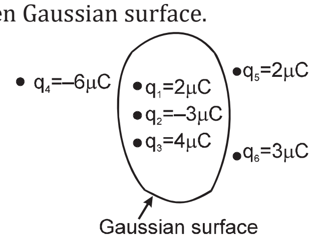
$$\phi_E = \oint \vec{E} \cdot \vec{dS} = \frac{1}{\varepsilon_0}\sum_{i=1}^{n} q_i$$

The circle on the sign of integration indicates that the integration is to be carried out over the closed surface.

Note:

1. Flux through Gaussian surface is independent of its shape.
2. Flux through Gaussian surface depends only on total charge present inside Gaussian surface.
3. Flux through Gaussian surface is independent of position of charges inside Gaussian surface.
4. Electric field intensity at the Gaussian surface is due to all the charges present inside as well as outside the Gaussian surface.
5. In a close surface incoming flux is taken negative while outgoing flux is taken positive, because $\hat{n}$ is taken positive in outward direction.
6. In a Gaussian surface $\phi = 0$ does not imply $E = 0$ at every point of the surface but $E = 0$ at every point implies $\phi = 0$

Ex. Find out flux through the given Gaussian surface.



Sol. $S = \frac{Q_{in}}{\varepsilon_0} = \frac{2\mu C - 3\mu C + 4\mu C}{\varepsilon_0} = \frac{3\times10^{-6}}{\varepsilon_0} Nm^2/C$

Ex. If a point charge q is placed at the center of a cube, then find out flux through any one surface of cube.

Sol. Flux through 6 surfaces= $\frac{q}{\varepsilon_0}$. Since all the surfaces are symmetrical

So, flux through one surface $= \frac{1}{6}\frac{q}{\varepsilon_0}$

Electric Flux Using Gauss' Law

Statement: The aggregate electric flux emanating from a sealed surface equals the enclosed charge divided by the permittivity.

Let's consider an area element dA on an arbitrary Gaussian surface as depicted.

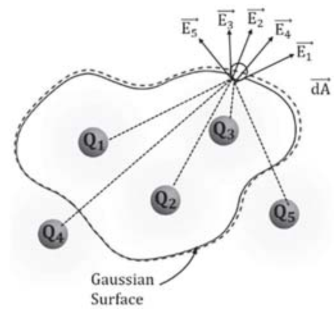
The resultant flux resulting from the field of multiple charges is...

$$d_{net} = \oint \vec{E}_{nut} \cdot \vec{dA}$$

$$= \oint (\vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4 + \vec{E}_5) \cdot \vec{dA}$$

$$d_{net} = \underbrace{\oint \vec{E}_1 \cdot \vec{dA}}_{\text{Flux due to } Q_1} \underbrace{\oint \vec{E}_2 \cdot \vec{dA}}_{\text{Flux due to } Q_2} \underbrace{\oint \vec{E}_3 \cdot \vec{dA}}_{\text{Flux due to } Q_3} \underbrace{\oint \vec{E}_4 \cdot \vec{dA}}_{\text{Flux due to } Q_4} \underbrace{\oint \vec{E}_5 \cdot \vec{dA}}_{\text{Flux due to } Q_5}$$

As the charges $Q4$ and $Q5$ are outside the closed surface, flux due to them through the closed surface is zero



Since the flux resulting from charges Q_4 and Q_5 equals zero, the overall flux...

$$Q_{net} = \frac{\theta_1}{\varepsilon_0} + \frac{\theta_2}{\varepsilon_0} + \frac{\theta_3}{\varepsilon_0} + 0 + 0 = \frac{\theta_1 + \theta_2 + \theta_3}{\varepsilon_0} = \frac{\theta_i}{\varepsilon_0}$$

$$\phi = \oint \vec{E}_{net} \cdot \vec{dA} = \frac{Q_{in}}{\varepsilon_0}$$

E_{net} originates from all charges, whether situated outside or within the Gaussian surface.
The overall flux arises solely from charges contained within the Gaussian surface.

$$\phi = \oint \vec{E}_{net} \cdot \vec{dA} = \frac{Q_{in}}{\varepsilon_0}$$

$$\phi = \oint \vec{E} \cdot \vec{dS}$$

Finding electric field from Gauss`s Theorem:

From gauss`s theorem, we can say

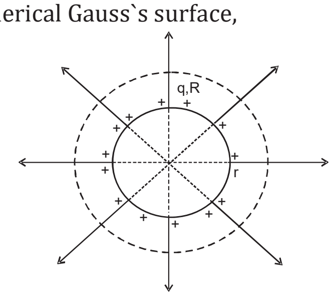
$$\int \vec{E} \cdot \vec{ds} = \phi_{net} = \frac{q_{in}}{\varepsilon_0}$$

Finding E due to a spherical shell:

Electric field outside the Sphere:

Since, electric field due to a shell will be radially outwards.
So let’s choose a spherical Gaussian surface Applying

Gauss`s theorem for this spherical Gauss`s surface,



$$\int \vec{E}\vec{ds} = \phi_{net} = \frac{q_{in}}{\varepsilon_o} = \frac{q}{\varepsilon_o}$$

↓

$\int |\vec{E}||\vec{ds}|\cos 0$ (because the $\vec{E}$ is normal to the surface)

↓

$E\int ds$ (because value of E is constant at the surface

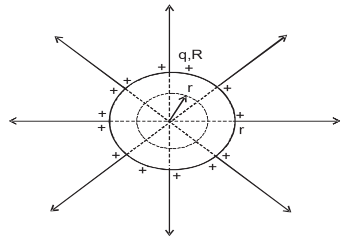
$E(4\pi r^2)(\int ds$ total area of the spherical surface $= 4\pi r^2)$

$$E(4\pi r^2) = \frac{q_{in}}{\varepsilon_0} \Rightarrow E_{out} = \frac{q}{4\pi\ {}_0 r^2}$$

Electric field inside a spherical shell:

Let's choose a spherical gaussian surface inside the shell.

Applying Gauss`s theorem for this surface



$$\int \vec{E}\vec{ds} = \phi_{net} = \frac{q_{in}}{\varepsilon_0} = 0$$

$$\downarrow$$

$$\int |\vec{E}||\vec{ds}|\cos 0$$

$$\downarrow$$

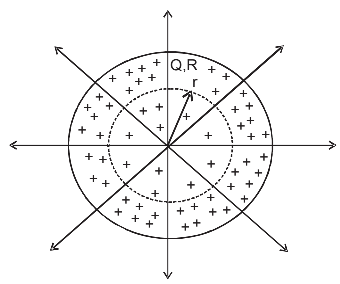
$$E\int ds$$

$$\downarrow$$

$$E(4\pi r^2) \Rightarrow E(4\pi r^2) = 0 \Rightarrow E_{in} = 0$$

Electric field inside a solid sphere:

For this choose a spherical Gaussian surface inside the solid sphere Applying gauss`s theorem for this surface.



$$\int \vec{E}\vec{ds} = \phi_{net} = \frac{q_{in}}{\varepsilon_0} = \frac{\frac{q}{\frac{4}{3}\pi R^3}\times\frac{4}{3}\pi r^3}{\varepsilon_0} = \frac{qr^3}{\varepsilon_0 R^3}$$

$$\downarrow$$

$$\int Eds$$

$$\downarrow$$

$$E(4\pi r^2) \Rightarrow E(4\pi r^2) = \frac{qr^3}{\varepsilon_0 R^3}$$

$$E = \frac{qr}{4\pi\varepsilon_0 R^3} \Rightarrow E = \frac{kQ}{R^3}r$$

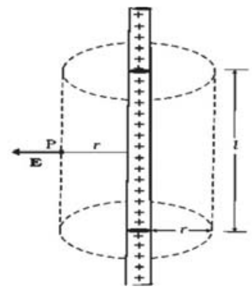
Applications Of Gauss's Law

The electric field due to a general charge distribution is, as seen above, In practice, except for some special cases, the summation (or integration) involved in this equation cannot be carried out to give electric field at every point in space. For some symmetric charge configurations, however, it is possible to obtain the electric field in a simple way using the Gauss's law. This is best understood by some examples.

Consider an infinitely long thin straight wire with uniform linear charge density λ. The wire is obviously an axis of symmetry. Suppose we take the radial vector from O to P and rotate it around the wire. The points P, P′, P′′ so obtained are completely equivalent with respect to the charged wire. This implies that the electric field must have the same magnitude at these points.

The direction of electric field at every point must be radial (outward if $\lambda > 0$, inward if $\lambda < 0$). This is clear consider a pair of line elements P_1 and P_2 of the wire, as shown. The electric fields produced by the two elements of the pair when summed give a resultant electric field which is radial (the components normal to the radial vector cancel). This is true for any such pair and hence the total field at any point P is radial. Finally, since the wire is infinite, electric field does not depend on the position of P along the length of the wire. In short, the electric field is everywhere radial in the plane cutting the wire normally, and its magnitude depends only on the radial distance r.

To calculate the field, imagine a cylindrical Gaussian surface, as shown in the (b). Since the field is everywhere radial, flux through the two ends of the cylindrical Gaussian surface is zero. At the cylindrical part of the surface, E is normal to the surface at every point, and its magnitude is constant, since it depends only on r. The surface area of the curved part is 2πrl, where l is the length of the cylinder.



Electric field due to an infinitely long thin straight wire is radial,

The Gaussian surface for a long thin wire of uniform linear charge density.

Flux through the Gaussian surface = flux through the curved cylindrical part of the surface.

$$E \times 2\pi rl$$

The surface includes charge equal to λ l. Gauss's law then gives $E \times 2\pi rl = \lambda l/\varepsilon$ oi.e.

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

Vector ally, E at any point is given by

$$E = \frac{\lambda}{2\pi \ {}_0 r}\hat{n}$$

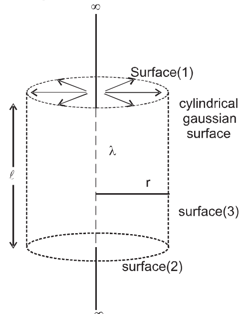
where n̂ is the radial unit vector in the plane normal to the wire passing through the point. E is directed outward if λ is positive and inward if λ is negative

Note that when we write a vector A as a scalar multiplied by a unit vector, i.e., as A = A aˆ, the scalar A is an algebraic number. It can be negative or positive.

The direction of A will be the same as that of the unit vector aˆ if $A > 0$ and opposite to aˆ if $A < 0$. When we want to restrict to non-negative values, we use the symbol A and call it the modulus of A. Thus, $A \geq 0$.

Also note that though only the charge enclosed by the surface (λl) was included above, the electric field E is due to the charge on the entire wire. Further, the assumption that the wire is infinitely long is crucial. Without this assumption, we cannot take E to be normal to the curved part of the cylindrical Gaussian surface. However, is approximately true for electric field around the central portions of a long wire, where the end effects may be ignored.

Electric field due to infinite line charge (having uniformly distributed charged of charge densityλ):



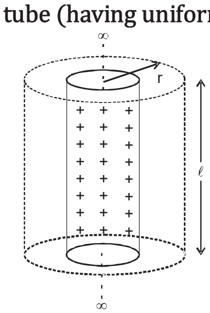
Electric field due to infinite wire is radial so we will choose cylindrical Gaussian surface as shown is figure.

$$\phi_{net} \rightarrow \phi_1 = 0 \quad \phi_2 = 0 \quad \phi_3 \neq 0 \; = \frac{q_{in}}{\varepsilon_0} = \frac{\lambda\ell}{\varepsilon_0}$$

$$\phi_3 = \int \vec{E} \cdot \vec{ds} = \int Eds = E\int ds = E(2\pi r\lambda)$$

$$E(2\pi r\lambda) = \frac{\lambda\ell}{\varepsilon_o} \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_o r} = \frac{2k\lambda}{r}$$

Electric field due to infinity long charged tube (having uniform surface charge density σ and radius R)):



1. **E outside the tube: -** lets choose a cylindrical Gaussian surface

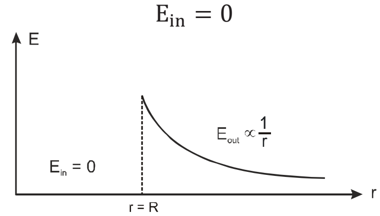
$$\phi_{net} = \frac{q_{in}}{\varepsilon_o} = \frac{\sigma 2\pi R\ell}{\varepsilon_o} \Rightarrow E_{out} \times 2\pi r\lambda = \frac{\sigma 2\pi R\ell}{\varepsilon_o} \Rightarrow E = \frac{\sigma R}{r\varepsilon_0}$$

2. **E inside the tube:**

Let's choose a cylindrical Gaussian surface inside the tube.

$$\phi_{net} = \frac{q_{in}}{\varepsilon_o} = 0$$

$$E_{in} = 0$$



E due to infinitely long solid cylinder of radius R (having uniformly distributed charge in volume (charge density ρ)):

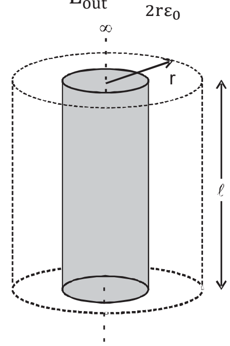
1. **E at outside point: -**

Let's choose a cylindrical Gaussian surface.

Applying gauss`s theorem

$$E \times 2\pi r\lambda = \frac{q_{in}}{\varepsilon_0} = \frac{\rho \times \pi R^2 \ell}{\varepsilon_0}$$

$$E_{out} = \frac{\rho R^2}{2r\varepsilon_0}$$



2. **E at inside point:**

Let's choose a cylindrical Gaussian surface inside the solid cylinder.

Applying gauss`s theorem $E \times 2\pi r\lambda = \frac{q_{in}}{\varepsilon_o} = \frac{\rho \times \pi r^2 \ell}{\varepsilon_o}$

$$E_{in} = \frac{\rho r}{2\varepsilon_0}$$