WAVE PARAMETERS

The wave equation is given as follows:

$$y = A\sin((\omega t - kx) + \phi)$$

1. **Angular frequency (ω)**

It refers to the rate of change of angular displacement of any element of a wave over time.

$$\omega = \frac{2\pi}{T}$$

Unit: Radian per second

2. **Angular wavenumber/Propagation constant (k)**

It represents the quantity of wavelengths within a given unit of distance. It is also referred to as spatial frequency.

$$k = \frac{2\pi}{\lambda}$$

Unit: Radian per meter

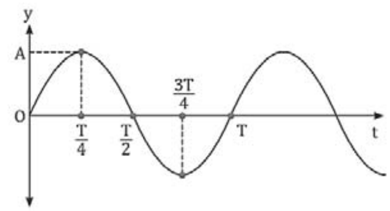
3. **Phase constant (ϕ)**

It delineates the extent of deviation of the waveform from its equilibrium position.

It signifies the alteration in phase per unit length along the wave's path at any given moment.

Unit: Radian per unit length

Let's examine the graph depicting displacement versus time of a wave traveling through a medium with n particles. As the wave traverses the medium, the particles execute Simple Harmonic Motion (SHM) around their average positions. Let's focus on a particular particle that was at its mean position at $t = 0$.



At $t = \frac{T}{4}$, At this point, the particle reaches its maximum displacement, causing its velocity to drop to zero.

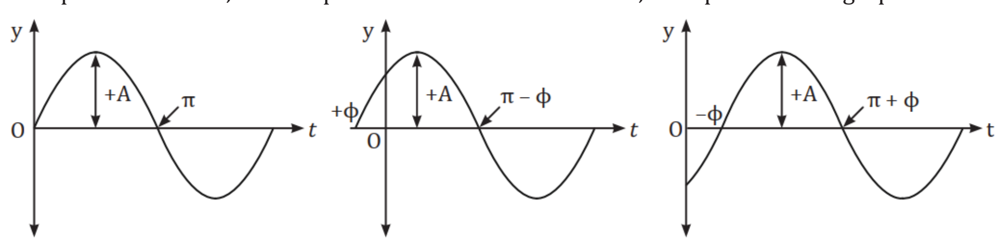
At $t = \frac{T}{2}$, The particle returns to its mean position and proceeds downward.

At $t = \frac{3T}{4}$ Once more, the particle reaches maximum displacement, but in the opposite direction. At this point, its velocity diminishes to zero, prompting a change in its direction.

At time $t = T$, the particle returns to its mean position, marking the completion of one oscillation.

For every specific time, there exists an angle associated with the position of the particle, termed as the particle's phase angle.

For a particle at $t = 0$, various phase values can be observed, as depicted in the graphs:



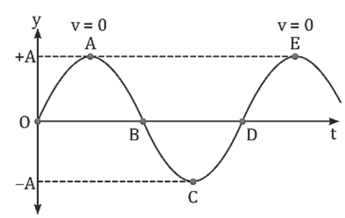
Positive initial phase constant: Sine curve starts from the left of the origin

$$y = A\sin(\omega t - kx + \phi)$$

Negative initial phase constant: Sine curve starts from the right of the origin.

$$y = A\sin(\omega t - kx - \phi)$$

We can note that particles at points A and E exhibit the same phase, indicating that their displacements from the mean position are identical. Particles in the same phase share the same displacement, velocity, acceleration, and energy due to the execution of Simple Harmonic Motion (SHM) around their mean positions within the medium. Following one complete time period T, they will once again synchronize in phase, and this pattern will persist.



Variation of Phase with Time

Two particles are in distinct phases, denoted as ϕ_{t_1} and ϕ_{t_2} respectively. The phase difference between them can be determined as follows:

$$y = A\sin(\omega t - kx + \phi)$$

The first phase is given by,

$$\phi_{t_1} = \omega t_1 - kx + \phi$$
$$\phi_{t_1} = \frac{2\pi}{T}t_1 - \frac{2\pi}{\lambda}x + \phi$$

The second phase is given by,

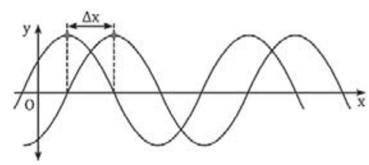
$$\phi_{t_2} = \omega t_2 - kx + \phi$$
$$\phi_{t_2} = \frac{2\pi}{T}t_2 - \frac{2\pi}{\lambda}x + \phi$$

The phase difference is given as follows:

$$\Delta\phi = |\phi_{t_2} - \phi_{t_1}|$$
$$\Delta\phi = (\frac{2\pi}{T}t_2 - \frac{2\pi}{\lambda}x + \phi) - (\frac{2\pi}{T}t_1 - \frac{2\pi}{\lambda}x + \phi)$$
$$\Delta\phi = \frac{2\pi}{T}(t_2 - t_1)$$
$$\Delta\phi = \frac{2\pi}{T}\Delta t$$

Phase difference $= \frac{2\pi}{T} \times$ Time difference

Variation of Phase with Distance



Let's examine two waves propagating along the x-direction with a time disparity of Δt, possessing phases ϕ_{x_1} and ϕ_{x_2}, respectively, as illustrated in the diagram.

It is the difference in the path traversed by two waves.

$$y = A\sin(\omega t - kx + \phi)$$

The first phase is given by,

$$\phi_{x_1} = \omega t - kx_1 + \phi$$
$$\phi_{x_1} = \frac{2\pi}{T}t - \frac{2\pi}{\lambda}x_1 + \phi$$

The second phase is given by,

$$\phi_{x_2} = \omega t - kx_2 + \phi$$
$$\phi_{x_2} = \frac{2\pi}{T}t - \frac{2\pi}{\lambda}x_2 + \phi$$

The phase difference is given as follows:

$$\Delta\phi = |\phi_{x_2} - \phi_{x_1}|$$
$$\Delta\phi = |(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x_2 + \phi) - (\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x_1 + \phi)|$$
$$\Delta\phi = \frac{2\pi}{\lambda}(x_2 - x_1)$$
$$\Delta\phi = \frac{2\pi}{\lambda}\Delta x$$

Phase difference $= \frac{2\pi}{\lambda} \times$ Path difference

Ex. What will be the path difference of two sound waves having a phase difference of 60° ?

(A) $\frac{\lambda}{6}$ (B) $\frac{\lambda}{3}$ (C) 2λ (D) $\frac{\lambda}{2}$

Sol. Given,

Phase difference, $\Delta\phi = 60°$

$$\Delta\phi = \frac{\pi}{3}\text{rad}$$

Also, $$\Delta\phi = \frac{2\pi}{\lambda}\Delta x$$
$$\frac{\pi}{3} = \frac{2\pi}{\lambda}\Delta x$$
$$\Delta x = \frac{\lambda}{6}$$

Thus, option (A) is the correct answer.

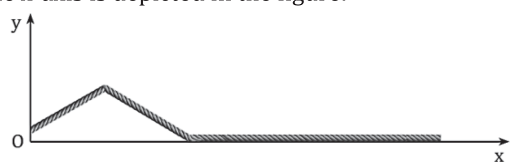
Travelling Wave

Imagine a scenario where a wave pulse is initiated in a string by an external source located at the origin, within a time interval from 0 to Δt. The pulse moves along a horizontal trajectory. A snapshot is captured within this interval, as illustrated in the figure. It is assumed that the source remains inactive before and after this time interval, meaning it does not induce any disturbances.

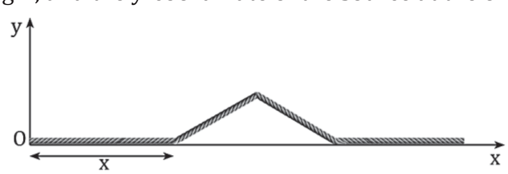
We can plot this scenario in the y-x graph for three time intervals as shown in the figures:

1. **For any time, $t < 0$:**
Throughout this interval, the source remains inactive, causing no disturbances along the y-axis. Consequently, the y-coordinate of the source is zero (located at the origin).

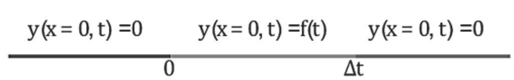
2. **For $0 < t < \Delta t$:**
During this interval, the source generates a disturbance, resulting in a wave pulse at the origin with a non-zero y-coordinate. Mathematically, this is expressed as $y(x = 0, t) = f(t)$. The pulse propagating along the x-axis is depicted in the figure:



3. **For any time, $t > \Delta t$:**
During this duration, the source does not generate any disturbance, resulting in the absence of a wave pulse at the origin, and the y-coordinate of the source at the origin remains zero.



If we represent the disturbance along the y-axis, denoted as f (t), for the time interval $0 < t < \Delta t$, the entire discussion can be illustrated in a figure as depicted:



Suppose the amplitude of the wave pulse is exceedingly small, and the string is uniform and homogeneous. It can be reasonably affirmed that the wave pulse will progress forward with a consistent velocity, denoted as v.

Hence, to traverse a distance x, the wave pulse will require a duration equivalent to $\frac{x}{v}$ thus it can be said that.

1. If at $t = 0$, the pulse is at $x = 0$, then, $t = \frac{x}{v}$, the pulse will be at $x = x$.
2. If at $t = t$, the pulse is at $x = 0$, then at, $t = t + \frac{x}{y}$, the pulse will be at $x = x$.
3. If at $t = t$, the pulse is at $x = x$, then at, $t = t - \frac{x}{v}$, the pulse will be at $x = 0$.

Therefore, it can be asserted that whatever occurs at $x = x$ at time t has already occurred at $x = 0$ at an earlier time. $t - \frac{x}{v}$ Mathematically, this can be written as follows:

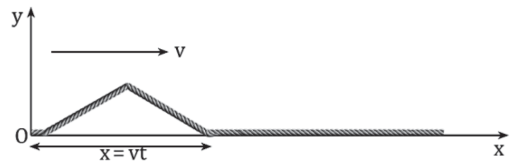
$$y(x, t) = y(x = 0, t - \frac{x}{v})$$

Consequently, if the equation of the source is known, we can deduce the complete wave equation.

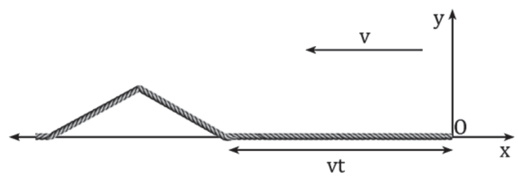
Given that $y(x=0,t) = f(t)$ represents the equation of the source at any given time t,

$$y(x,t) = y(x=0, t-\frac{x}{v}) = f(t-\frac{x}{v}) = g(x-vt)$$

If an equation is in the form of $f(t-\frac{x}{v})$ or $g(x-vt)$ then the equation represents a travelling wave in the positive x-direction.



If a given equation takes the form of $f(t+\frac{x}{v})$ or $g(x+vt)$ then the equation represents a travelling wave in the negative x-direction.



Definitions

1. **Travelling wave:**

Any mathematical expression or formula, either in existence or subject to manipulation as $g(x \pm vt)$ or $f(t \pm \frac{x}{v})$ represents a travelling wave.

2. **Wave functions:**

Which depict the waves, are termed as such. The wave functions for traveling waves with velocity v are $g(x \pm vt)$ and $f(t \pm \frac{x}{v})$

3. **Phase:**

The term $(x \pm vt)$ is referred to as the phase of the wave function. In the case of a wave pulse, the phase remains constant as the pulse's shape remains unchanged. Essentially, the phase is synonymous with the shape of the wave pulse.

4. **Phase velocity:**

The defined quantity for the velocity of the phase in a traveling wave is $v = \frac{dx}{dt}$ This is also known as wave velocity.

Ex. A wave is propagating on a long, stretched string along its length taken as the positive x-axis. The wave equation is given as, $y = y_0 e^{-(\frac{t}{T}-\frac{x}{\lambda})^2}$ where, $y_0 = 4$ cm, T = 1 s, and $\lambda = 4$ cm. Find the following:

(a) Velocity of the wave.

(b) Function f (t) giving the displacement of the particle at x = 0.

(c) Function g(x) giving the shape of the string at t = 0.

(d) Plot the shape g(x) of the string at t = 0.

(e) Plot the shape g(x) of the string at t = 5 s.

Sol. Let us rewrite the given wave equation as follows:

$$y = y_0 e^{-(\frac{t}{T}-\frac{x}{\lambda})^2}$$

$$y = y_0 e^{-\frac{1}{T^2}(t-\frac{xT}{\lambda})^2}$$

$$y = y_0 e^{-\frac{1}{T^2}(t-\frac{x}{(\frac{\lambda}{T})})^2} \quad \dots (1)$$

(a) Equation (i) is in the form of $f(t-\frac{x}{v})$ Therefore, the provided equation signifies a traveling wave. Upon comparing the structures of the phase components, we obtain:

$$v = \frac{\lambda}{T}$$

$$v = \frac{4\text{ cm}}{1\text{s}}$$

$$v = 4\text{ cms}^{-1}$$

Alternate We have, $y = y_0 e^{-\frac{1}{T^2}(t-\frac{x}{(\frac{\lambda}{T})})^2}$

Therefore, the phase of the travelling wave is, $(t - \frac{x}{(\frac{\lambda}{T})})$ and we know that the phase of the travelling wave remains constant.

$$\frac{d}{dt}(t - \frac{x}{(\frac{\lambda}{T})}) = 0$$

$$1 - \frac{T}{\lambda}\frac{dx}{dt} = 0$$

$$\frac{dx}{dt} = \frac{\lambda}{T}$$

$$v = \frac{\lambda}{T}$$

$$v = \frac{4cm^2}{1s}$$

$$v = 4\ cms^{-1}$$

(b) By putting $x = 0$ in equation (1), we get,

$$y = y_0 e^{-\frac{1}{T^2}(t-0)^2}$$

$$y = y_0 e^{-(\frac{t}{T})^2}$$

$$f(t) = y_0 e^{-(\frac{t}{T})^2}$$

(c) By putting $t = 0$ in equation (1), we get,

$$y = y_0 e^{-(0-\frac{x}{\lambda})^2}$$

$$y = y_0 e^{-(\frac{x}{\lambda})^2}$$

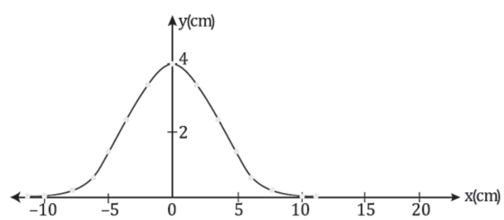
$$g(x) = y_0 e^{-(\frac{x}{\lambda})^2}$$

(d) We know that at $t = 0, g(x) = y_0 e^{-(\frac{x}{\lambda})^2}$

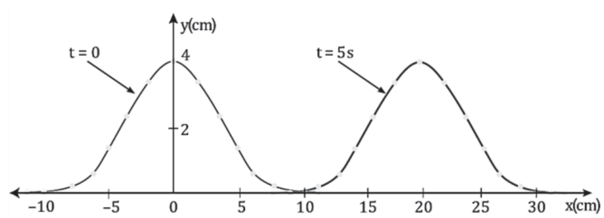
Now, by putting $x = 0$, we get,

$$g(x = 0) = y_0 = 4\ cm$$

Therefore, the peak of the graph will be at x = 0 with g(x = 0) = 4 cm. The graph is as shown in the figure.



(e) At $t = 5s, g(x) = y_0 e^{-(5-\frac{x}{\lambda})^2}$ and the graph is as following:



Ex. A sinusoidal wave is travelling along a rope. The oscillator that generates the wave completes 60 vibrations in 30 s. Also, a given pulse travels 425 cm along the rope in 10 s. What is the wavelength?

Sol. We know that frequency is defined as the number of oscillations per unit time. Therefore, the frequency is $f = \frac{60}{30} = 2$ Hz

Given, The wave pulse travels 425 cm along the rope in 10 s. Hence, the wave velocity is,

$$v = \frac{425}{10} = 42.5\ cms^{-1}$$

Thus, the wavelength is, $\lambda = \frac{v}{f} = \frac{42.5}{2} = 21.25cm$

Ex. A wave pulse is travelling on a string at 2 ms^{-1}. Displacement y of the particle at x = 0 at any time t is given by $y = \frac{2}{t^2+1}$. Find the following:

(a) Expression of function y = (x, t), i.e., the displacement of a particle at position x and time t.

(b) Shape of the pulse at t = 0 and t = 1 s.

Sol. Given, The displacement of the particle at x = 0 at any time t is, $y = \frac{2}{t^2+1}$

The velocity of the wave pulse is, v = 2 ms^{-1}.

(a) We know that if y(x = 0, t) = f (t) is the equation of the source at any time t, then the displacement of the particle at position x and time t is as follows:

$$y(x,t) = y(x=0, t-\frac{x}{v})$$

$$y(x,t) = \frac{2}{(t-\frac{x}{v})^2+1}$$

$$y(x,t) = \frac{2}{(t-\frac{x}{2})^2+1}$$

(b) Now we have, $y(x,t) = \frac{2}{(t-\frac{x}{2})^2+1}$

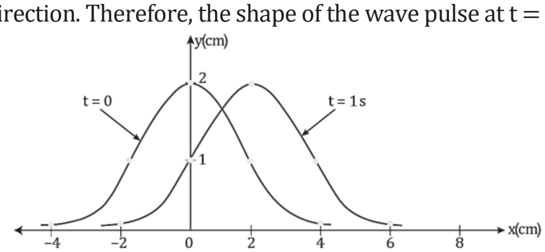
By putting t = 0, we get, $y(x, t = 0) = \frac{2}{\frac{x^2}{4}+1}$ ... (1)

And by putting t = 1s, we get,

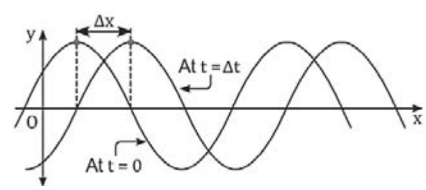
$$y(x, t = 1) = \frac{2}{(1-\frac{x}{2})^2+1}$$

$$y(x, t = 1) = \frac{2}{\frac{(x-2)^2}{4}+1} \quad \text{... (2)}$$

From equations (1) and (2), it is seen that there is a difference of 2 m between the peaks of the wave pulses along the x-direction. Therefore, the shape of the wave pulse at t = 0 and t = 1 s is as follows:



Velocity of a Sinusoidal Wave



Let's consider a wave propagating along the x-direction at two different times, t = 0 and t = Δt, with corresponding phases ϕ_{x_1} and ϕ_{x_2} as depicted in the illustration. It's understood that the displacement of a wave can be expressed generally as follows:

$$y = A \sin(\omega t - kx)$$

To get the velocity of the wave, we have to differentiate displacement equation

$$\sin(\omega t - kx) = \text{Constant}$$

$$\omega t - kx = \text{Constant}$$

Differentiating with respect to time, we get the following:

$$\omega\frac{dt}{dt} - k\frac{dx}{dt} = 0$$

$$\omega - k\frac{dx}{dt} = 0$$

$$\frac{dx}{dt} = \frac{\omega}{k}$$

$$v = \frac{\omega}{k}$$

Where, V = Velocity of the wave

As we know, wave function of a wave is, $y = f(ax \pm bt)$,

$$\text{Velocity of the sinusoidal wave } = \frac{\text{Coefficient of t}}{\text{Coefficient of x}}$$

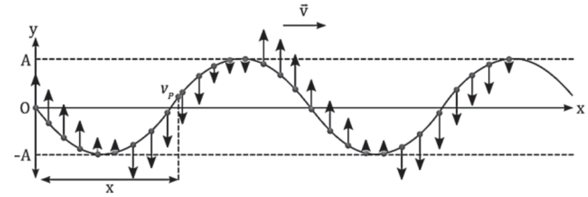
From the equation of displacement of wave,

$$y = f(ax + bt) = (\omega t - kx)$$

Hence, the velocity of a sinusoidal wave $= \frac{\text{Coefficient of t}}{\text{Coefficient of x}} = \frac{\omega}{k}$

Speed of a Particle in a Sinusoidal Wave

Let's examine a wave traveling in the x-direction and observe a particle located at a distance x from the origin.



We are aware that every particle constituting the wave must adhere to the general equation governing the wave's displacement.

$$y = A \sin(\omega t - kx)$$

Where, x is constant as we are focusing only on one particle.

The transverse speed of the particle in the sinusoidal wave is as follows:

$$v_P = \frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(A\sin(\omega t - kx))$$
$$v_P = \omega A\cos(\omega t - kx)$$

The acceleration of the particle is given as follows:

$$a_P = \frac{\partial}{\partial t}(v_P)$$
$$a_P = -\omega^2 A\sin(\omega t - kx)$$
$$a_P = -\omega^2 y (\because y = A\sin(\omega t - kx))$$

Hence, by examining the acceleration of a wave, we can deduce that the particles within the medium are undergoing Simple Harmonic Motion (SHM) along the y-direction, while the wave progresses in the positive x-direction. Now, considering all particles at any given moment, i.e., at constant time,

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x}(A\sin(\omega t - kx))$$
$$\frac{\partial y}{\partial x} = -kA\cos(\omega t - kx)$$
$$\frac{\partial y}{\partial x} = \frac{-k}{\omega}\frac{\partial y}{\partial t}(\because v_P = \omega A\cos(\omega t - kx))$$
$$\frac{\partial y}{\partial x} = -\frac{1}{v}\frac{\partial y}{\partial t}\left(\because v = \frac{\omega}{k}\right)$$
$$\frac{\partial y}{\partial t} = -v\frac{\partial y}{\partial x}$$

$v_P = -$ Wave velocity $\times$ Slope of the wave curve

General Wave Equation

$$y = A\sin(\omega t - kx)$$

Differentiating with respect to x,

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x}(A\sin(\omega t - kx))$$
$$\frac{\partial y}{\partial x} = -kA\cos(\omega t - kx)$$

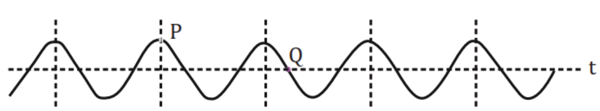
Differentiating again with respect to x,

$$\frac{\partial^2 y}{\partial x^2} = -k^2 A\sin(\omega t - kx)$$
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}\left(\because a_y = \frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(\omega t - kx)\right)$$
$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial y}{\partial x^2}$$

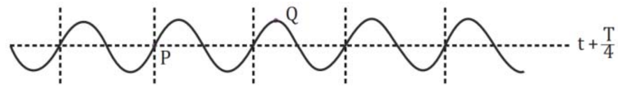
This is the general wave equation.

Picturisation of Wave Equation

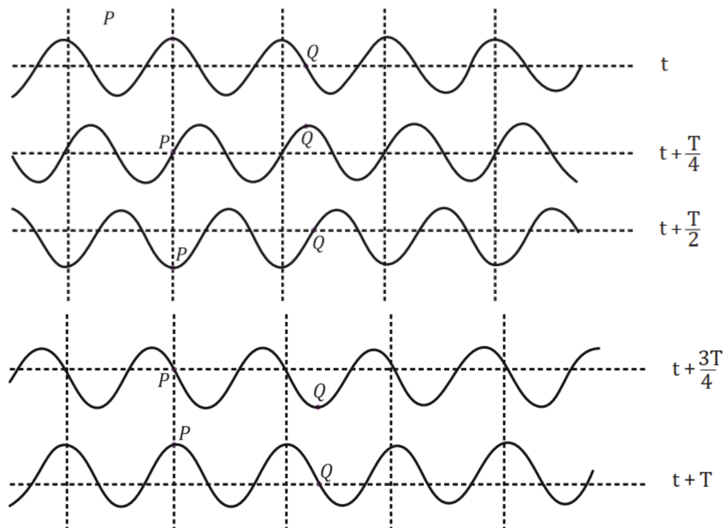
Contemplate two particles, P and Q, located at any given time t on a wave propagating along a string. This wave takes on the shape of a sinusoidal wave, resulting from the Simple Harmonic Motion (SHM) of the particle at x = 0 (source undergoing SHM), as illustrated in the figure.



Since we understand that the smallest indivisible unit of time for Simple Harmonic Motion (SHM) is $\frac{T}{4}$ (T is the time period of the source), at time $t = \frac{T}{4}$ the locations of particles P and Q will be as follows:



By this way, if we simultaneously increase the time by $\frac{T}{4}$ then the corresponding positions of both the particles will be as follows:

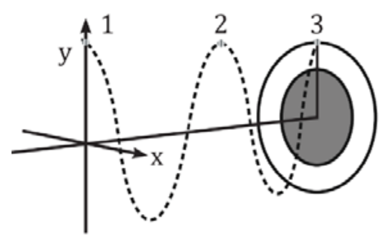


Upon close examination of the image, it can be observed that particles P and Q return to their initial positions after one time period T. Consequently, one can infer that each particle on the string undergoes Simple Harmonic Motion (SHM) with the same amplitude and time period as the source. In summary, every particle on the string wave exhibits identical characteristics to those of the source. Thus, the wave equation reflects the equation of the source undergoing SHM.

Phase

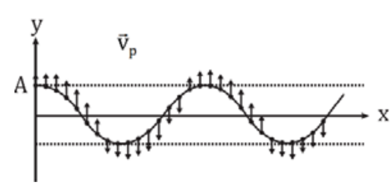
The general expression of the wave equation is, $y = A \sin(\omega t - kx + \phi)$

In mathematical terms, the total instantaneous phase of the wave is represented by the argument of the sine function, denoted as $(\omega t - kx + \phi)$, where ϕ is the epoch of the source. Nevertheless, the phase is commonly considered a wave parameter, and the wave repeats itself after every 2π phase value. A phase shift of $2n\pi$ results in points oscillating in phase, meaning that the behavior of each particle remains the same, where n is an integer. In the illustration, points 1, 2, and 3 mark the positions of wave repetition.



Velocity and acceleration of a particle

Assume the overall expression for the displacement of a particle on the string wave or sinusoidal wave at a specific x-coordinate is provided by,



$$y_p = A\sin(\omega t - kx + \phi)$$

Then, the velocity of the particle is,

$$v_p = [\frac{dy}{dt}]_{x=\text{Constant}} = \frac{\partial y}{\partial t} = \omega A\cos(\omega t - kx + \phi)$$

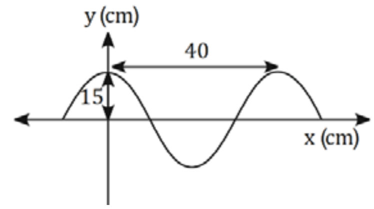
And the acceleration of the particle is,

$$a_p = [\frac{d^2y}{dt^2}]_{x=\text{Constant}} = \frac{\partial^2 y}{\partial t^2}$$

$$a_p = -\omega^2 A\sin(\omega t - kx + \phi)$$

$$a_p = -\omega^2(y_p)$$

Ex. A sinusoidal wave travelling in the positive x-direction has an amplitude of 15 cm, a wavelength of 40 cm, and a frequency of 8 Hz. The vertical displacement of the medium at $t = 0$ and $x = 0$ is also 15 cm.



(a) Find the angular wave number, period, angular frequency, and speed of the wave.

(b) Determine the phase constant (ϕ) and write a general expression for the wave function.

Sol. Given, Amplitude of the wave, $A = 15$ cm

Wavelength of the wave, $\lambda = 40$ cm

Frequency of the wave, $f = 8$ Hz

The vertical displacement of the medium at $t = 0$ and $x = 0$ is, $y(x = 0, t = 0) = 15$ cm

Also, it is given that the wave is sinusoidal and is travelling in the positive x-direction.

(a) The angular wave number of the wave is, $k = \frac{2\pi}{\lambda} = \frac{2\pi}{40} = \frac{\pi}{20}\ \text{cm}^{-1}$

The time period of the wave is, $T = \frac{1}{f} = \frac{1}{8}$ s

The angular frequency of the wave is, $\omega = 2\pi f = (2\pi) \times 8 = 16\pi s^{-1}$ (the unit of angular frequency is rad s^{-1}, but here, π is expressed in rad; so, the unit is written as s^{-1} only.)

The speed of the wave is, $v = \frac{\omega}{k} = \frac{16\pi}{(\frac{\pi}{20})} = 320\text{cms}^{-1}$

(b) Since the vertical displacement of the medium at $t = 0$ and $x = 0$ is 15 cm and from the figure, it is seen that this is the positive extreme position. As the particle starts from the positive extreme position, the phase constant or the epoch is $\frac{\pi}{2}$ rad.

Since the sinusoidal wave is travelling in the positive x-direction, the general expression of the wave function is, $y = A\sin(\omega t - kx + \phi)$

By substituting the values of the wave parameters, we get,

$$y = (15\text{ cm})\sin[(16\pi s^{-1})t - (\frac{\pi}{20}\text{cm}^{-1})x + \frac{\pi}{2}]$$

$$y = (15\text{ cm})\cos[(16\pi s^{-1})t - (\frac{\pi}{20}\text{cm}^{-1})x]$$