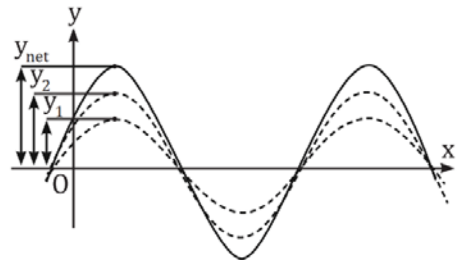
SUPERPOSITION OF WAVES – INTERFERENCE

Principle of Superposition

If various external agents concurrently induce distinct disturbances, leading to diverse vertical displacements (or amplitudes) in the same string, the resultant vertical displacement in the string will be the vector sum of the individual displacements.

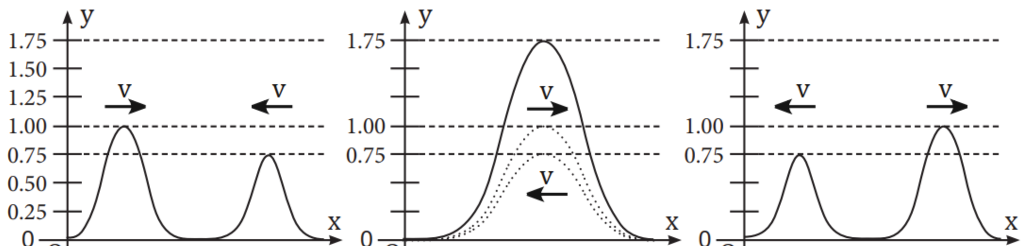


Mathematically, If $\vec{y}_1, \vec{y}_2, \vec{y}_3, \ldots\ldots, \vec{y}_n$ If the individual displacements in the string are given as, then the resultant vertical displacement will be,

$$\vec{y}_{net} = \vec{y}_1 + \vec{y}_2 + \vec{y}_3 + \cdots \ldots \ldots \ldots + \vec{y}_n = \sum \vec{y}_i$$

Case 1 (Two wave pulses have positive amplitude and move along the same plane):

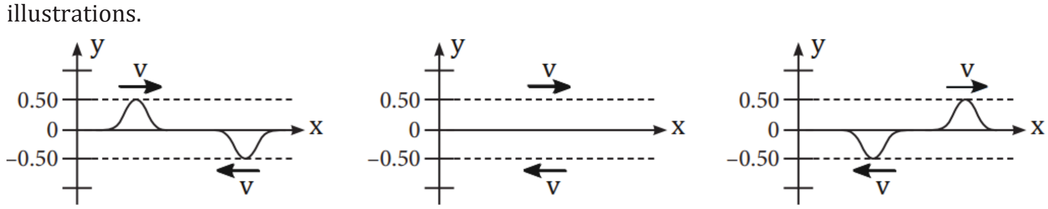
Imagine two wave pulses traveling in opposite directions, each exhibiting positive vertical displacements. $\vec{y}_1$ and $\vec{y}_2$, respectively, then at the moment of superposition the resultant displacement becomes $\vec{y}_{net} = \vec{y}_1 + \vec{y}_2 = |\vec{y}_1| + |\vec{y}_2|$ as show in the following figure.



Any moment before the superposition | **At the superposition** | **Any moment after the superposition**

Case 2 (Both the pulses move along the same plane, where one has a negative amplitude and the other has a positive amplitude):

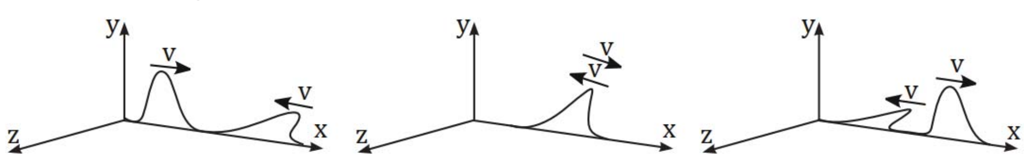
In this case, at the moment of superposition, the resultant displacement becomes, $\vec{y}_{net} = \vec{y}_1 + \vec{y}_2 = |\vec{y}_1| - |\vec{y}_2|$ as depicted in the accompanying figure. In this case, if the vertical displacements of the two wave pulses attain the same magnitude, the net displacement becomes zero. This implies that the pulses cancel each other at the moment of superposition, a scenario supported by the provided illustrations.



Any moment before the superposition | **At the superposition** | **Any moment after the superposition**

Case 3 (Two pulses move along two planes perpendicular to each other with two different amplitudes):

In this scenario, at the moment of superposition, the resulting amplitude becomes, $y_{net} = \sqrt{y_1^2 + y_2^2}$, and the resulting pulse forms an angle of 45° with the planes in which the individual pulses were moving.



Any moment before the superposition | **At the superposition** | **Any moment after the superposition**

Note:

1. The pulses that emerge after the overlap are indistinguishable from the original pulses.
2. The waves can pass through each other.
3. If $\vec{y}_1, \vec{y}_2, \vec{y}_3, \ldots \ldots, \vec{y}_n$ If simultaneous individual displacements are provided in the string, then the resultant vertical displacement in the string is expressed as:

$$\vec{y}_{net} = \vec{y}_1 + \vec{y}_2 + \vec{y}_3 + \cdots \ldots + \vec{y}_n = \sum_i \vec{y}_i$$

Difference between Particle Nature and Wave Nature

	Particle nature		Wave nature
1	In the realm of particle physics, energy is conveyed via the exchange of mass.	1	In the domain of wave characteristics, energy is conveyed without the need for mass transfer.
2	In the realm of particle characteristics, the outcome following the collision between two particles is entirely altered.	2	In wave behavior, following the superposition (analogous to a collision for waves) of two or more wave pulses, the waveform profile remains unchanged from its state prior to superposition.

Ex. Two waves passing through a region are represented as follows: $y_1 = (5\text{ mm}) \sin[(2\pi\text{ cm}^{-1})x - (50\pi\text{ s}^{-1})t]$ and $y_2 = (10\text{ mm}) \sin[(\pi\text{ cm}^{-1})x - (100\pi\text{ s}^{-1})t]$ Find the displacement of the particle at x = 1 cm at time t = 5.0 ms.

Sol. Given

$$y_1 = (5\text{ mm})\sin[(2\pi\text{cm}^{-1})x - (50\pi\text{s}^{-1})t]. \quad \ldots (1)$$

$$y_2 = (10\text{ mm})\sin[(\pi\text{cm}^{-1})x - (100\pi\text{s}^{-1})t] \quad \ldots (2)$$

By putting x = 1 cm and t = 5.0 ms = 5.0×10^{-3} s in equations (1) and (2), we get,

$$y_1 = (5\text{ mm})\sin[(2\pi - (50\pi\text{s}^{-1})(5.0 \times 10^{-3}\text{ s})]$$

$$y_1 = (5\text{ mm})\sin\left(2\pi - \frac{\pi}{4}\right)$$

$$y_1 = -(5\text{ mm})\sin\left(\frac{\pi}{4}\right)\left[\text{Since }\left(2\pi - \frac{\pi}{4}\right)\text{ lies on }4^{\text{th}}\text{ quadrant, }\sin\left(2\pi - \frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right)\right]$$

$$y_1 = -\frac{5}{\sqrt{2}}\text{mm}$$

Now, $$y_2 = (10\text{ mm})\sin[(\pi\text{cm}^{-1})(1\text{cm}) - (100\pi\text{s}^{-1})(5.0 \times 10^{-3}\text{ s})]$$

$$y_2 = (10\text{ mm})\sin\left(\pi - \frac{\pi}{2}\right)$$

$$y_2 = (10\text{ mm})\sin\left(\frac{\pi}{2}\right)$$

$$y_2 = 10\text{ mm}$$

Therefore, the resultant displacement of the particle at x = 1 cm at time t = 5.0 ms is given by

$$y_{net} = y_1 + y_2$$

$$y_{net} = 10 - \frac{5}{\sqrt{2}}\text{ mm}$$

Interference

The occurrence of the simultaneous overlay of two or more waves with the described characteristics is referred to as wave interference.

The properties are as follows:

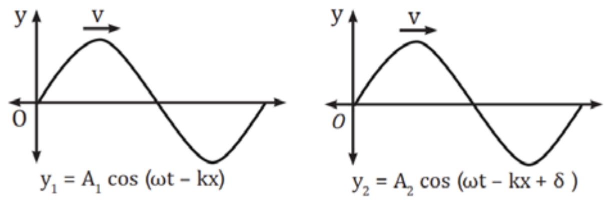
1. The waves must propagate in parallel directions.
2. The waves must possess identical wavelengths.
3. The waves must exhibit matching frequencies.
4. A consistent phase difference between the waves must be maintained.

Coherent waves:

Waves with a consistent phase difference between them are termed coherent waves, and the originator of such waves is referred to as a coherent source.

Ex. $y_1 = A_1\cos(\omega t - kx)$ and $y_2 = A_2\cos(\omega t - kx + \delta)$ These two waves are coherent.

Suppose the superposition of two coherent waves is as follows:

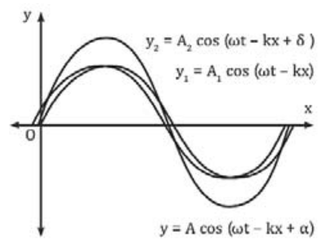


$$y_1 = A_1\cos(\omega t - kx)$$
$$y_2 = A_2\cos(\omega t - kx + \delta)$$

This corresponds to the superposition of two simple harmonic motions (SHMs) in the identical direction and along the same line. Hence, we anticipate a comparable outcome to what we observed with the superposition of SHMs.

Hence, the equation for the resulting wave is expressed as:

$$y = A\cos(\omega t - kx + \alpha)$$
$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\delta} \text{ and}$$
$$\tan\alpha = \frac{A_2\sin\delta}{A_1 + A_2\cos\delta}$$



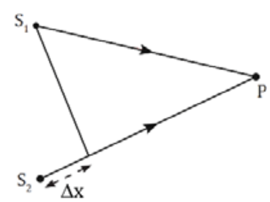
Thus, it can be inferred that the interaction between two sinusoidal coherent traveling waves propagating in the same direction results in the emergence of another sinusoidal coherent traveling wave moving in the identical direction.

Incoherent source

Incoherent sources refer to sources that generate waves with varying frequencies, wavelengths, and phases.

Conditions for interference

1. Two sinusoidal waves must exhibit coherence.
2. They need to propagate in the same direction.



Consider two sources, S_1 and S_2, emitting waves that pass through a point P. It's evident that the wave emitted from source S_2 must travel a longer distance compared to the wave from source S_1. This variance in the path traversed by the waves leads to a phase difference.

Let the equation of wave from first source S_1 be the following:

$$y_1 = A_1\sin(\omega t - kx)$$

The equation of wave from source S_2 is as follows:

$$y_2 = A_2\sin(\omega t - kx + \phi)$$

From the principle of superposition, we get,

$$\vec{y}_{net} = \vec{y}_1 + \vec{y}_2 + \vec{y}_3 + \cdots .. + \vec{y}_n$$

In this case, $y_{net} = y = y_1 + y_2$

$$y = A_1\sin(\omega t - kx) + A_2\sin(\omega t - kx + \phi)$$
$$y = A_1\sin(\omega t - kx) + A_2[\sin(\omega t - kx)\cos\phi + \cos(\omega t - kx)\sin\phi]$$
$$y = (A_1 + A_2\cos\phi)\sin(\omega t - kx) + A_2\cos(\omega t - kx)\sin\phi \quad \text{... (1)}$$

Let $(A_1 + A_2\cos\phi) = A\cos\alpha$ ... (2)

Let $(A_1 + A_2\cos\phi) = A\cos\alpha$

And, $A_2\sin\phi = A\sin\alpha$ ... (3)

By substituting these values in equation (1), we get the following:

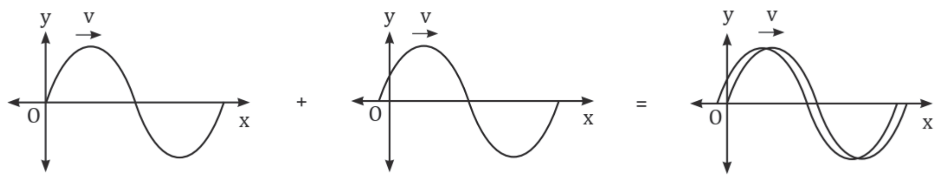
$$y = A\cos\alpha\sin(\omega t - kx) + A\sin\alpha\cos(\omega t - kx)$$
$$y = A[\cos\alpha\sin(\omega t - kx) + \sin\alpha\cos(\omega t - kx)]$$
$$y = A\sin(\omega t - kx + \alpha)\ (\because \sin(A + B) = \sin A\cos B + \cos A\sin B)$$

By squaring and adding equations (2) and (3), we get,

$$A^2\cos^2\alpha + A^2\sin^2\alpha = (A_1 + A_2\cos\phi)^2 + (A_2\sin\phi)^2$$
$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$$

By dividing equation (3) by equation (2), we get the following:

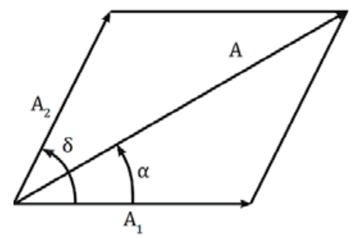
$$\tan(\alpha) = \frac{A\sin\alpha}{A\cos\alpha} = \frac{A_2\sin\phi}{A_1 + A_2\cos\phi}$$



$y_1 = A_1 \sin(\omega t - kx)$ $y_2 = A_2 \sin(\omega t - kx + \varphi)$ $y = y_1 + y_2$, $y = A \sin(\omega t - kx + \alpha)$

Geometrical interpretation

The magnitudes of two distinct waves and their resultant wave can be illustrated on a phasor diagram akin to vector addition diagrams. Let's take two phasors, A_1 and A_2, with an angle δ between them. These phasors can be combined as vectors. The phasor diagram is depicted in the nearby figure.



The amplitude of the resultant wave is given by

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\delta}$$

Therefore, it can be asserted that the magnitude of the resultant wave amplitude is contingent upon the phase disparity between the individual waves.

The phase angle of the resultant wave is given by

$$\alpha = \tan^{-1}\left(\frac{A_2\sin\delta}{A_1 + A_2\cos\delta}\right)$$

Ex. Two sinusoidal waves of the same frequency travel in the same direction along a string. If $A_1 = 3.0$ cm, $A_2 = 4.0$ cm, $\phi_1 = 0$, and $\phi_2 = \frac{\pi}{2}$ rad then what is the amplitude of the resultant wave?

Sol. The amplitude of the resultant wave is given by,

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\delta}$$

The phase difference between the given waves is as follows:

$$\delta = \phi_2 - \phi_1 = \frac{\pi}{2}\text{rad}$$

By substituting all the given values in this equation, we get,

$$A = \sqrt{3^2 + 4^2 + 2 \times 3 \times 4 \times \cos\frac{\pi}{2}}$$

$$A = \sqrt{3^2 + 4^2}$$

$$A = \sqrt{9 + 16}$$

$$A = \sqrt{25} = 5 \text{ cm}$$

Therefore, the amplitude of the resultant wave is 5 cm.

Special Cases of Interference

Case 1: $\cos\delta = 1$ (Constructive interference)

For $\cos\delta = 1$,

$$\cos\delta = \cos(2n\pi)$$

$$\delta = 2n\pi$$

Where n = Integer = 0,1,2,3 …

Hence, when the phase difference is an even multiple of π, constructive interference occurs.

In this scenario, the amplitude of the resulting wave is given by:

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\delta}$$

$$A = \sqrt{A_1^2 + 2A_1A_2 + A_2^2}$$

$$A = A_1 + A_2$$

This represents the maximum attainable amplitude for the resulting wave. Thus, in constructive interference, the resultant wave achieves the following amplitude:

$$A_{max} = A_1 + A_2$$

Ex. Let's examine the two sinusoidal coherent traveling waves as follows:

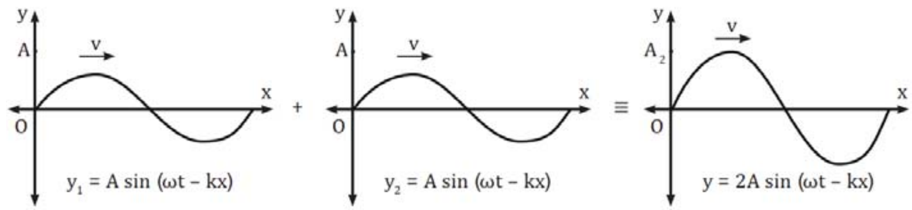
$$y_1 = A\sin(\omega t - kx)$$

$$y_2 = A\sin(\omega t - kx)$$

Here, the phase difference between them is zero. It means that $\cos\delta = 1$. Therefore, constructive interference takes place and the resultant wave becomes,

$$y = 2A\sin(\omega t - kx)$$

Observe that the amplitude of the resulting wave is simply the sum of the amplitudes of the individual waves.



Case 2: $\cos\delta = -1$ (Destructive interference)

For $\cos\delta = -1$,

$$\cos\delta = \cos[(2n+1)\pi]$$
$$\delta = (2n+1)\pi$$

Where n = Integer = 0,1,2,3 ...

Hence, when the phase difference is an odd multiple of π, destructive interference occurs.

As a result, the amplitude of the resulting wave becomes.

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\delta}$$
$$A = \sqrt{A_1^2 - 2A_1A_2 + A_2^2}$$
$$A = |A_1 - A_2|$$

This represents the minimum achievable amplitude for the resulting wave. Thus, in destructive interference, the resultant wave achieves the following amplitude:

$$A_{min} = |A_1 - A_2|$$

Ex. Let's examine two sinusoidal coherent traveling waves as follows:

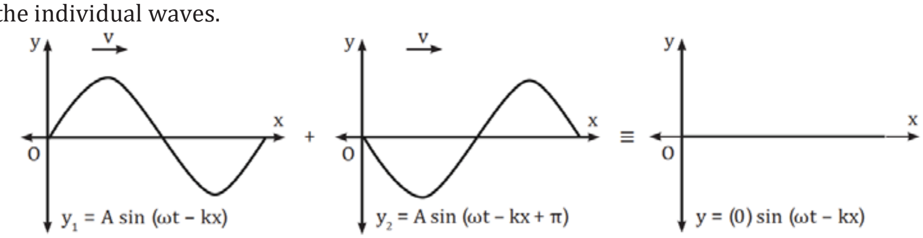
$$y_1 = A\sin(\omega t - kx)$$
$$y_2 = A\sin(\omega t - kx + \pi)$$

In this case, the phase difference between them is π, indicating that $\cos\delta$ equals -1. Consequently, destructive interference occurs, resulting in the following resultant wave:

$$y = (0)\sin(\omega t - kx)$$

This is the wave with zero amplitude.

Note that the amplitude of the resulting wave is simply the difference between the amplitudes of the individual waves.



Interference

In the previous session, we have learnt about constructive and destructive interference. We will learn more about it in this session.

Intensity of the resultant wave

It is understood that the amplitude of the resulting wave is determined by

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\delta}$$

Now, remember that in a given medium, the intensity of a wave is directly proportional to the square of its amplitude.

Thus, the intensity of the resultant wave becomes

$$I_{net} \propto A^2$$
$$I_{net} = cA^2 \text{ Where c is the proportionality constant]}$$
$$I_{net} = c[A_1^2 + A_1^2 + 2A_1A_2\cos\delta]$$
$$I_{net} = [(cA_1^2) + (cA_2^2) + 2cA_1A_2\cos\delta]$$

Considering that the intensity of each individual wave is likewise proportional to the square of its amplitude, the equation transforms into:

$$I_{net} = I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2}\cos\delta \quad \dots (1)$$

If the intensity of the individual wave becomes equal, i.e.,
$I_1 = I_2 = I$, then equation (i) becomes,

$$I_{net} = I + I + 2\sqrt{I}\sqrt{I}\cos\delta$$
$$I_{net} = 2I + 2I\cos\delta$$
$$I_{net} = 2I(1 + \cos\delta)$$
$$I_{net} = 2I[2\cos^2(\frac{\delta}{2})][\text{ Since, } 1 + \cos\delta = 2\cos^2(\frac{\delta}{2})]$$
$$I_{net} = 4I\cos^2(\frac{\delta}{2}) \quad \dots (2)$$

For constructive interference

Constructive interference occurs when the phase difference between the interfering waves becomes an even multiple of π, denoted as δ = 2nπ, where n is an integer (n = 0, 1, 2, 3...). In such instances, the intensity of the resulting wave is given by:

$$I_{net} = I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2} \text{ [By putting } \delta = 2n\pi \text{ or } \cos\delta = 1 \text{ in equation (1)]}$$

Therefore, the maximum intensity becomes,

$$I_{max} = I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2}$$
$$I_{max} = (\sqrt{I_1} + \sqrt{I_2})^2$$

Now, if $I_1 = I_1 = I$, then,
$I_{net} = 4I$ [We can also prove this by putting $\delta = 0$ in equation (2).]

For destructive interference

Destructive interference occurs when the phase difference between the interfering waves becomes an odd multiple of π, represented as δ = (2n + 1) π, where n is an integer (n = 0, 1, 2, 3…). In such instances, the intensity of the resulting wave is:

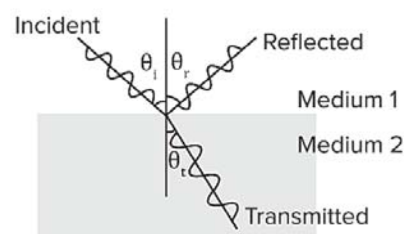
$$I_{net} = I_1 + I_2 - 2\sqrt{I_1}\sqrt{I_2} \text{ [By putting } \delta = (2n+1)\pi \text{ or } \cos\delta = -1 \text{ in equation (1)]}$$

Therefore, the minimum intensity becomes,

$$I_{max} = I_1 + I_2 - 2\sqrt{I_1}\sqrt{I_2}$$
$$I_{max} = (\sqrt{I_1} - \sqrt{I_2})^2$$

Now, if $I_1 = I_2 = I$, then from equation (2), we get,
$I_{net} = 0$ [We can also prove this by putting $\delta = \pi$ in equation (2).]

Wave at an Interface



Imagine a sinusoidal traveling wave encountering the boundary between two media, both of which are lossless, meaning they cannot absorb the waves. Upon incidence, a portion of the incident wave reflects back into the original medium while another portion transmits into the second medium, as illustrated in the diagram below:

If the frequency and velocity of the incident, reflected, and transmitted waves are denoted as f, f', f'', and v, v', v'' respectively, then it holds that f = f' = f'' and v = v' ≠ v'' because the frequency of waves depends solely on the source, whereas the velocity of waves is a characteristic property of the medium.

Intensity of the Wave

The wave intensity is defined as the 'average quantity of energy passing through a cross-sectional area per unit time' or the 'average power conveyed per unit area perpendicular to the wave's propagation direction.'

Mathematically, it is defined as follows:

$$\text{Intensity (I)} = \frac{\text{Average power } (P_{avg})}{\text{Cross-sectional area (S)}}$$
$$I = \frac{2\pi^2 f^2 A^2 \mu v}{S}$$
$$I = \frac{2\pi^2 f^2 A^2 \rho S v}{S}$$

[Since we know that, Linear mass density (μ) = Volume mass density (ρ) $\times$ S]

$$I = 2\pi^2 f^2 A^2 \rho v$$

For a particular traveling wave within a defined medium, the values of frequency (f), density (ρ), and velocity (v) remain constant. Therefore, it can be concluded that: $I \propto A^2$. This also implies that $\langle P_{ava} > \propto A^2$.

Note:

1. The average power propagated along the string during the wave transmission is expressed as $P_{avg} = 2\pi^2 f^2 A^2 \mu v = \frac{\omega^2 A^2 F}{2v}$

2. The wave's intensity is expressed as $I = 2\pi^2 f^2 A^2 \rho v$

3. For a specific travelling wave in a known medium, (Intensity) $\propto$ (Amplitude)^2 and also, (Average power transmitted) $\propto$ (Amplitude)^2 .