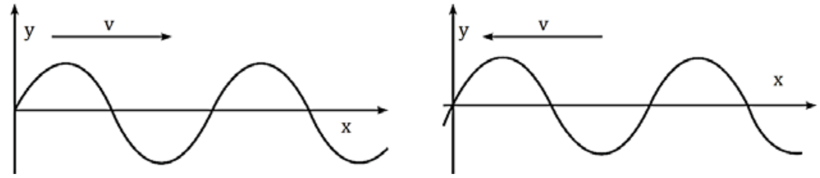
STANDING WAVE ON A STRING

Introduction to Standing Waves

Imagine two sinusoidal coherent waves traveling in opposite directions with identical wave parameters. Let one wave propagate along the positive x-direction and the other along the negative x-direction.



Therefore, the equations of these waves are given by.

$$y_1 = A\sin(kx - \omega t) \quad \dots (1)$$

$$y_2 = A\sin(kx - \omega t) \quad \dots (2)$$

When superposition occurs between these two waves, the equation for the resultant wave is given by:

$$y = y_1 + y_2$$

$$y = A\sin(kx - \omega t) + A\sin(kx + \omega t)$$

$$y = A[\sin(kx - \omega t) + \sin(kx + \omega t)]$$

Since we know that, $\sin C + \sin D = 2\sin(\frac{C+D}{2})\cos(\frac{C-D}{2})$ we get the following,

$$y = A[2\sin(\frac{(kx-\omega t)+(kx+\omega t)}{2})\cos(\frac{(kx-\omega t)-(kx+\omega t)}{2})]$$

$$y = 2A\sin(kx)\cos(-\omega t)$$

As cosine is an even function, $\cos(-x) = \cos x$

Therefore, the expression becomes

$$y = 2A\sin(kx)\cos(\omega t) \quad \dots (1)$$

When two sinusoidal traveling waves superimpose, the resulting wave takes the form of another traveling wave, and its equation is:$f(t \pm \frac{x}{v})$ However, in this scenario, it's not possible to manipulate the equation of the resulting wave into the form $f(t \pm \frac{x}{v})$ so, the resultant wave is not a travelling wave. If we denote $2A\sin(kx) = A(x)$, then the equation takes the following form:

$$y = A(x)\cos(\omega t) \quad \dots (2)$$

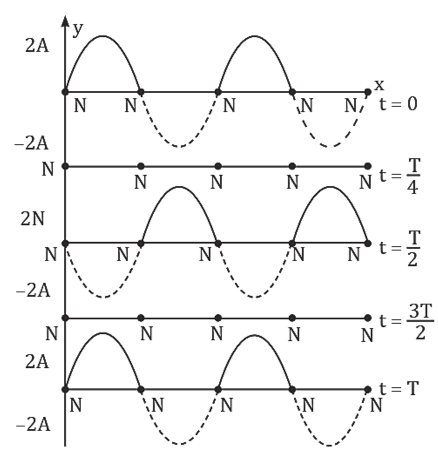
This equation signifies the cumulative motion of particles undergoing Simple Harmonic Motion (SHM) with identical angular frequency. However, the amplitude of these particles varies according to their respective positions.

Now, substituting $kx = n\pi$ (where $n = 0, 1, 2,$) into equation (1), we obtain:

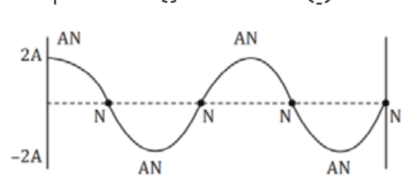
$$y = 2A\sin(n\pi)\cos(\omega t)$$

Let $n = 0, \Rightarrow y = 0$ (for any time t)

So, for each, $x = \frac{n\pi}{k} = \frac{n\pi}{(\frac{2\pi}{\lambda})} = \frac{n\lambda}{2},$



At all times, the particles undergoing Simple Harmonic Motion experience a displacement along the y-direction, which is equal to zero. This implies that the particles located at $x = \frac{n\lambda}{2}$ or $x = 0, \frac{\lambda}{2}, \lambda, \dots$ etc. These points undergo Simple Harmonic Motion with zero amplitude continuously. They are illustrated in the provided figure. The points undergoing Simple Harmonic Motion with the smallest amplitude (zero) are referred to as nodes, while those experiencing SHM with the maximum amplitude are termed antinodes. In the provided figure, N and AN designate the nodes and antinodes, respectively.



As the particles at nodes experience zero displacement continuously, they are incapable of transferring energy. Consequently, between two consecutive nodes, energy becomes trapped or confined. It's essential to note that while energy remains confined between the nodes, The total mechanical energy oscillates between maximum kinetic and potential energy, owing to the Simple Harmonic Motion of each particle.

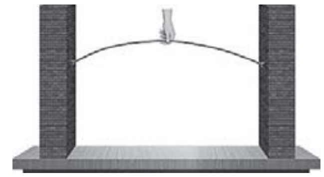
Definition:

When two harmonic waves with identical frequencies and amplitudes propagate in opposite directions within a medium and overlap, they generate a standing or stationary wave.

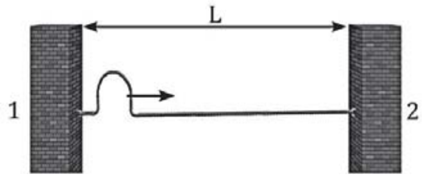
Ex: Standing waves are generated by the strings of a guitar.

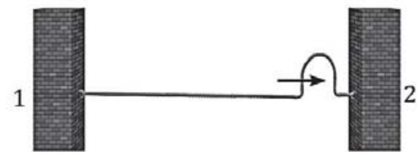
Standing Wave in a String Fixed at Both Ends

Imagine a string that is anchored at both ends, and an external force has plucked the string somewhere between the fixed ends, as depicted in the figure.

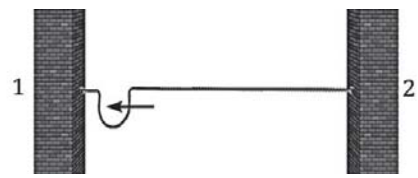


Now, let's imagine that the external agent initiates a wave pulse at the fixed end 1 of the string, causing it to propagate along the length of the string.

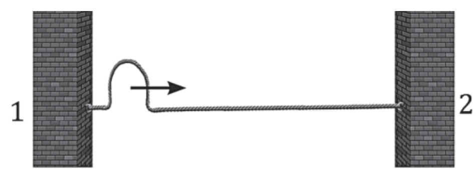




Upon reaching the fixed end 2, the pulse undergoes reflection, causing its profile to invert. Subsequently, the inverted pulse travels back towards fixed end 1 and undergoes reflection from fixed end 1.



Following another round of reflection, the pulse once again undergoes inversion and eventually regains its original shape after a phase change of 2π or a path difference of 2L. This cycle persists until there is no dissipation at the ends. This scenario is depicted in the adjoining figure.



If the external agent continues to generate wave pulses, the pulse that has already traversed the path difference of 2L and reverted to its original shape will superimpose with the pulse created by the external agent. Consequently, constructive interference occurs, leading to an increase in the amplitude of the resultant wave. However, due to the presence of dissipative forces at the ends, equilibrium is eventually established, and the amplitude stabilizes at a specific value.

Condition for constructive interference

Let's assume the length of the string is L. Consequently, the reflected wave travels a distance of 2L, which surpasses the distance covered by the initial pulse generated at the starting end. Consequently, the path difference between them amounts to 2L.

We understand that for constructive interference to occur between these two waves, the phase difference should be a multiple of 2π, expressed as $2n\pi$ (where n is an integer, including 0, 1, 2 ...). Additionally, we're aware that a path difference of λ between the waves corresponds to a phase difference of 2π. Hence, in a broader context, for any path difference Δx, the phase difference becomes: $\Delta\phi = \frac{2\pi}{\lambda}\Delta x$

Therefore, the phase difference between those two waves is given by,

$$\Delta\phi = \frac{2\pi}{\lambda}\Delta x = \frac{2\pi}{\lambda}(2L)$$

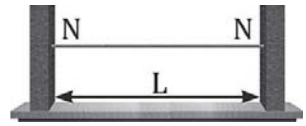
Hence, for constructive interference, the required condition is given by,

$$\frac{2\pi}{\lambda}(2L) = 2n\pi$$

$$\frac{2L}{\lambda} = n$$

$$2L = n\lambda.. \quad ...(1)$$

By examining equation (i), it becomes apparent that if we substitute n with values such as 0, 1, 2... We can determine the positions of nodes, as L takes on different values. $0, \frac{\lambda}{2}, \lambda$ For each For values of n ranging from 0, 1, 2... The fixed ends serve as nodes due to the string being clamped at both ends.



Hence, when a string is fixed at both ends, the incident and reflected waves interact to create a stationary transverse wave, with the ends always acting as nodes.
Let's consider the equation of a standing wave given by:

$$y = 2A\sin(kx)\cos(\omega t) \quad ...(2)$$

As nodes are present at both fixed ends of the string, the boundary conditions are as follows:

(a) At $x = 0, y = 0$ for time t (b) At $x = L, y = 0$ for time t

By applying condition (b) in equation (2), we get,

$$A\sin(kL)\cos(\omega t) = 0$$

If this equation evaluates to zero due to the 'sine' function, we obtain:

$$\sin(kL) = 0$$

$$\sin(kL) = \sin(n\pi)$$

$$kL = n\pi$$

$$\frac{2\pi}{\lambda}L = n\pi$$

$$\frac{2L}{\lambda} = n$$

$$2L = n\lambda$$

This equation is identical to equation (i). Hence, the requirement to generate a perfect standing wave on a string fixed at both ends is expressed as:

$$2L = n\lambda$$

$$L = \frac{n\lambda}{2} \quad ...(3)$$

Therefore, to establish a flawless standing wave on a string secured at both ends, the length of the string should be an integer multiple of $\frac{\lambda}{2}$. If v and f are If v and f represent the velocity and frequency, respectively, of the two parent waves whose superposition forms the standing wave, then another condition can be expressed as follows:

$$2L = n\lambda$$

$$2L = n\left(\frac{v}{f}\right)\left[\text{ Since, Wavelength } (\lambda) = \frac{\text{Velocity (v)}}{\text{Frequency (f)}}\right]$$

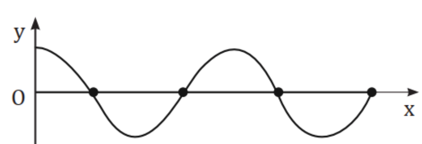
$$f = n\left(\frac{v}{2L}\right) \quad ...(4)$$

Therefore, it can be concluded that not all frequencies induce a flawless standing wave on a string secured at both ends; rather, specific frequencies possess this capability, and they are termed resonant frequencies. These resonant frequencies engender various modes of vibrations.

Energy of a Standing Wave

Particle velocities reach their maximum at the mean positions, indicating maximum kinetic energy and zero potential energy. Consequently, the total energy equals the kinetic energy. Conversely, at extreme positions, potential energy is maximized while kinetic energy becomes zero. Therefore, the total energy equals the potential energy.

At mean position



At extreme position

Energy of a standing wave in one loop

The wave particles undergo Simple Harmonic Motion (SHM) with varying amplitudes based on their positions (x). Let's focus on one loop of a standing wave and examine an element of mass dm with a thickness dx, located at a distance x from the origin of the loop. Considering that kinetic energy (KE) reaches its maximum at the mean position,

$$TE = KE_{max}$$
$$d(KE_{max}) = \frac{1}{2} \times dm \times v_{p,max}^2$$
$$\mu = \frac{dm}{dx}$$
$$dm = \mu dx$$

And, $y = -2\,A\sin kx\cos \omega t$

$$v_p = \frac{\partial y}{\partial t} = 2A\omega \sin kx\sin \omega t$$
$$v_{p,max} = 2A\omega \sin kx$$
$$v_{p,\,max}^2 = 4A^2\omega^2 \sin^2 kx$$
$$d(KE_{max}) = \frac{1}{2}\mu dx \times 4A^2\omega^2 \sin^2 kx$$
$$d(TE) = 2A^2\omega^2\mu \sin^2 kxdx$$
$$\int_0^{TE} d\,(TE) = 2A^2\omega^2\mu \int_0^{\frac{\lambda}{2}} \sin^2\, kxdx$$
$$TE = 2A^2\omega^2\mu \int_0^{\frac{\lambda}{2}} \frac{[1-\cos 2kx]}{2}\,dx$$
$$TE = A^2\omega^2\mu[x - \frac{\sin 2kx}{2k}]_0^{\frac{\lambda}{2}}$$
$$TE = \frac{1}{2}\lambda A^2\omega^2\mu$$