REFLECTION AND TRANSMISSION IN A STRING

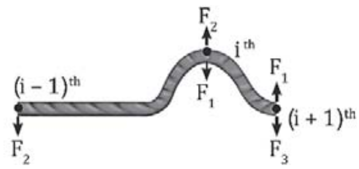
Reflection of a Wave

The reflection of a wave comprises two parts:

1. Reflection from the fixed end **2.** Reflection from the free end

1. Reflection from the fixed end

Imagine a string anchored at one end. At the opposite end, an external force is applied to generate a wave pulse. Typically, the anchored end of the string is referred to as a knot. Now, envision the moment just before the wave pulse reaches the knot, depicted in the figure below.

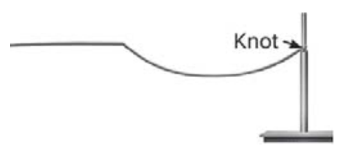


Now, the inquiry arises: What happens to the wave pulse upon hitting the fixed end? This can be comprehended through the following examination:

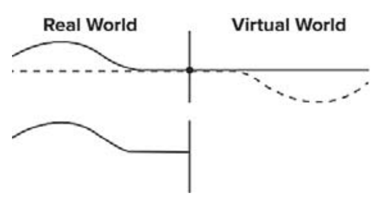
Take into account three particles, denoted as,$(i-1)^{th}$, i^{th}, and $(i+1)^{th}$, within the wave pulse on the string, as illustrated in the figure.

As the wave pulse advances, it appears that the particle at the crest of the pulse, namely the ith particle, exerts an upward force (F_1) on the subsequent particle, i.e., the $(i+1)^{th}$ particle, and conversely, the same force F_1 acts on the ith particle in the downward direction as a reaction. Likewise, focusing on the $(i-1)^{th}$ particle, which was previously at the crest, it can be envisioned that the $(i-1)^{th}$ particle applies an upward force (F_2) on its neighboring particle, i.e., the ith particle, and correspondingly, the same force F_2 acts on the $(i-1)^{th}$ particle in the downward direction as a reaction. Given that the i^{th} particle is at the crest, the resultant force ($F_1 - F_2$) on it ought to be directed downward, necessitating an upward force on the $(i+1)^{th}$ particle, which will momentarily occupy the crest position.

Likewise, when the $(i+1)^{th}$ particle ascends to the crest, it imparts an upward force (F_3) to its subsequent counterpart, experiencing an equal and opposite downward force (F_3) in reaction. Consequently, at that moment, the net downward force on the $(i+1)^{th}$ particle equals ($F_3 - F_1$), while its successor perceives an upward force. We have addressed the scenarios for all particles in the string except the terminal one, which is affixed to a wall. For the last particle, any upward force exerted on it by the penultimate particle is countered by an equal downward force from the wall. Hence, the net force on the last particle of the string is nullified. It's noteworthy that as the penultimate particle exerts an upward force on the last particle, there ensues a net downward reaction force on it.

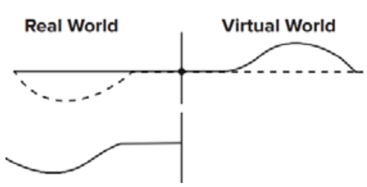


Hence, it can be inferred that the net force on the terminal particle is neutralized due to the wall, while there exists a resultant downward force on the penultimate particle. Consequently, the shape of the string's wave pulse undergoes inversion following reflection from the fixed end.



This entire scenario can be succinctly summarized, as depicted in the figure.

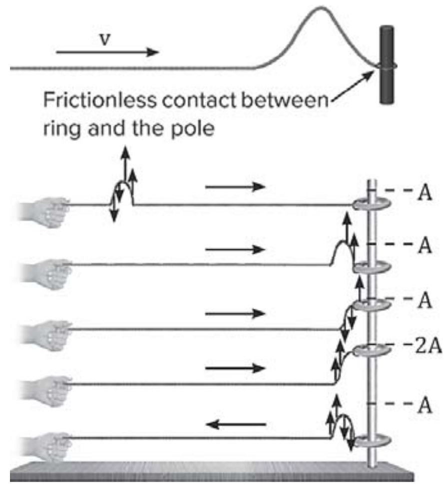
Visualize a real-world scenario where the original wave pulse propagates, alongside a hypothetical scenario where an identical wave pulse with an opposite phase commences moving in the opposite direction of the original wave pulse, as illustrated in the figure.



The waves from these two distinct worlds intersect at the knot, with the wave pulse from the hypothetical world entering the real world, and vice versa. Despite this intersection, they preserve their original shapes, as the superposition of the two waves does not alter the forms of the wave pulses.

Hence, it can similarly be deduced that the shape of the pulse during reflection can be determined by overlaying the inverted pulse onto the incident pulse.

Reflection from the free end

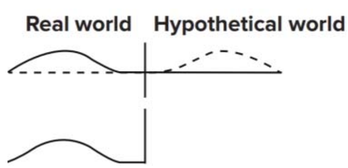


Imagine a frictionless ring attached to one end of a string and a wave pulse being generated by an external force applied at the opposite end.

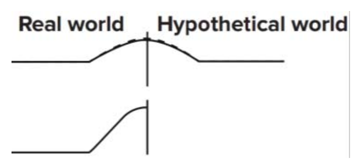
In this instance, the net force acting on the final particle of the string isn't nullified since the wall or pole lacks the capacity to exert the necessary force, owing to a lightweight, frictionless ring attached to the string. Consequently, there exists a resultant upward force on the terminal particle of the string. Consider a wave pulse with an amplitude of A approaching the pole. At the moment the pulse reaches the pole, the ring ascends to a height of 2A due to the upward force acting on the terminal particle (in accordance with energy conservation principles), and subsequently returns to its original position. Consequently, the pulse is reflected without undergoing inversion of its shape. The entire discussion is illustrated in the figure.

Now, considering the perspectives of the real and hypothetical worlds, it can be envisaged as two wave pulses originating from distinct realms, each with the same phase, converging towards the pole and superimposed upon each other. Essentially, this convergence accounts for the upward movement of the ring to a height of 2A. Subsequently, the wave pulse from the hypothetical world transitions into the real world, while the wave pulse from the real world enters the hypothetical world. Notably, their interaction does not alter each other's shape. The entirety of this scenario is illustrated in the figure.

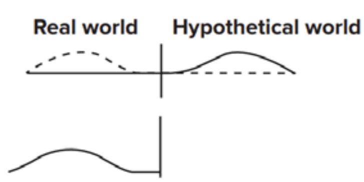
Before superposition of wave pulses of same phase moving in opposite direction



At the moment of superposition of wave pulses of same phase moving in opposite direction



Hence, the shape of the string's wave pulse remains unaltered after reflection from the free end. It can also be inferred that in this scenario, determining the shape of the pulse during reflection involves superimposing the same pulse onto the incident pulse.

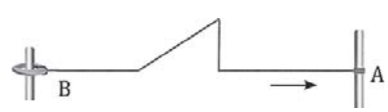


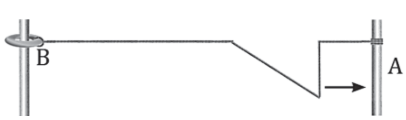
After superposition, of wave pulses of same phase moving in opposite direction

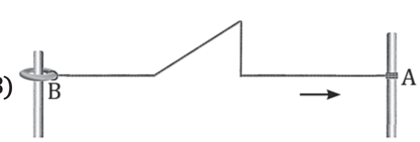
Note:

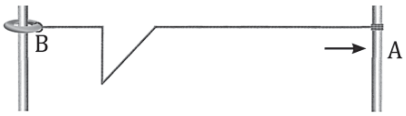
1. The shape of the string wave pulse becomes inverted upon reflection from the fixed end.
2. The shape of the string wave pulse remains unchanged after reflection from the free end.

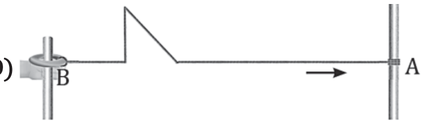
Ex. The pulse depicted reflects first from the rigid wall, labeled A, and subsequently from the free end, marked as B.



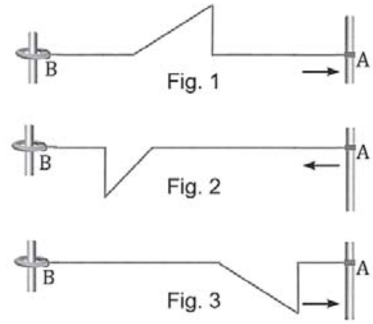
(A) 

(B) 

(C) 

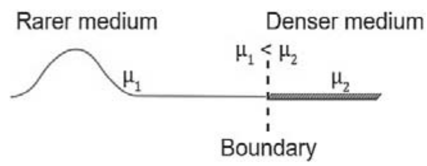
(D) 

Sol. The initial shape of the wave pulse is, is given in Fig. 1.
It can be easily observed that point A is a fixed end and point B is a free end.
We know that the wave pulse gets inverted after reflecting from the fixed end, and the pulse retains its incident form after reflecting from the free end. Therefore, after getting reflected from point A, the pulse gets the shape as shown in Fig. 2.
After getting reflected from point B, the pulse gets the shape as shown in Fig. 3.
Thus, option (A) is the correct answer.



Wave Propagation between Two Strings

Reflecting waves from fixed and free ends represent idealized concepts. Let's delve into a practical scenario. Imagine a situation where a lightweight string with linear mass density μ_1 is connected to a heavier string with linear mass density μ_2. Here, $\mu_1 < \mu_2$, hence the lightweight string can be regarded as the less dense medium, while the heavier string serves as the denser medium.



This scenario occurs when the knot or the boundary between the two strings is neither fully fixed nor entirely free. Consequently, there is partial reflection alongside partial transmission.

As the velocity of a wave in a medium is determined by $v = \sqrt{\frac{F}{\mu}}$ in this scenario, where the force applied is equal for both strings, the velocity of the wave in the less dense medium surpasses that of the denser medium. The denser medium, characterized by the lower wave velocity, is designated as having a fixed boundary, while the less dense medium, exhibiting the higher wave velocity, is identified with a free boundary.

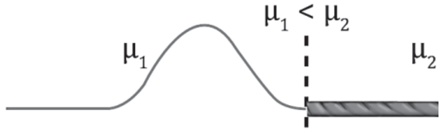
It's important to note that the comparison of density between two mediums is entirely contingent on the specific type of wave involved. For example, let's consider an interface between air and water. When a light wave transitions from air to water, its velocity decreases. However, when a sound wave moves from air to water, its velocity increases. This indicates that water is optically denser but acoustically less dense than air.

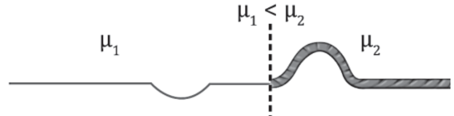
Returning to the discussion concerning the light and heavy strings, we can conclude that:

- The transmitted wave never undergoes inversion.
- If the wave is traveling on the light string (characterized by high wave velocity) and reflects from the heavy string (characterized by low wave velocity, akin to a fixed end), then the reflected wave undergoes inversion.

Ex. **Incident wave is on light string:**

When the incident wave propagates on the light string and reflects from the heavy string, the reflected wave undergoes inversion, while the transmitted wave remains unaffected by inversion.

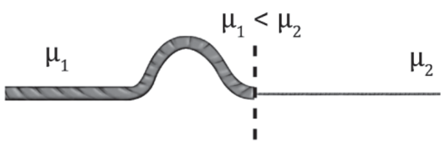


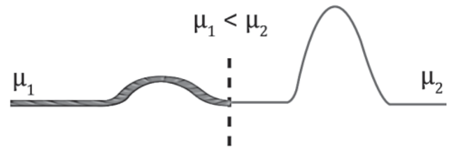


When the wave travels through the heavy string (characterized by low wave velocity) and reflects from the light string (characterized by high wave velocity, similar to a free end), the reflected wave preserves its original incident waveform.

Ex. **Incident wave is on heavy string:**

When the incident wave occurs on the heavy string and reflects from the light string, the reflected wave remains uninvited, and the transmitted wave also remains uninvited.





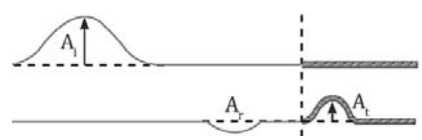
The medium characterized by low wave velocity serves as the denser medium and corresponds to the fixed end. Conversely, the medium with high wave velocity represents the rarer medium and corresponds to the free end. This relationship is illustrated in the table below:

Reflected from	Wave inversion
Low wave velocity medium	Yes
High wave velocity medium	No
↓	
Reflected from	**Wave inversion**
Fixed end	Yes
Free end	No
↓	
Reflected from	**Wave inversion**
Denser medium	Yes
Rarer medium	No

Did you observe the difference in amplitude between the reflected and transmitted waves in both examples? Would you like an explanation as to why the amplitudes of the reflected and transmitted waves differ? Let's delve into this in the following section.

Amplitude of Reflected Wave and Transmitted Wave

Suppose the incident wave occurs on the light string (the rarer medium), with an amplitude denoted as Ai. Let's assume that the speed of the incident wave and the transmitted wave is v_1 and v_2, respectively. If the incident wave is transmitted to the heavy string (the denser medium), then $v_1 > v_2$.



Consider the amplitude of the reflected wave and transmitted wave as A_r and A_t, respectively. At the boundary, a portion of the incident wave is reflected, and another portion is transmitted. If we consider the propagation direction of the incident wave along the positive x-direction, we can express the equation as follows:

$$y_i = A_i \sin[\omega(t - \frac{x}{v_1})]$$

Given that the incident wave travels along the positive x-direction, the reflected and transmitted waves propagate along the negative x-direction and the positive x-direction, respectively. Hence, the equation for the reflected wave is expressed as:

$$y_r = A_r \sin[\omega(t + \frac{x}{v_1})]$$

The equation of the transmitted wave is given by,

$$y_t = A_t \sin[\omega(t - \frac{x}{v_2})]$$

From the conservation of energy, it can be said that the average power of the incident wave gets divided into reflected and transmitted waves. Therefore,

$$\langle P_i \rangle = \langle P_r \rangle + \langle P_t \rangle \quad \text{... (1)}$$

Since we know that the average power transmitted by a wave in a medium is $P = \frac{F\omega^2 A^2}{2v}$,

From equation (1), we get,

$$\frac{F\omega^2 A_i^2}{2v_1} = \frac{F\omega^2 A_r^2}{2v_1} + \frac{F\omega^2 A_t^2}{2v_2}$$

$$\frac{A_i^2}{v_1} = \frac{A_r^2}{v_1} + \frac{A_t^2}{v_2} \quad \text{... (1)}$$

Now, at the right of the boundary between the two strings, the amplitude is A_t and at the left of the boundary, the amplitude is, $A_i - A_r$. Therefore,

$$A_i - A_r = A_t \quad \text{... (3)}$$

From equation (2), we get,

$$\frac{A_i^2}{v_1} = \frac{A_r^2}{v_1} + \frac{A_t^2}{v_2}$$

$$\frac{A_i^2}{v_1} - \frac{A_r^2}{v_1} = \frac{A_t^2}{v_2}$$

$$\frac{A_i^2 - A_r^2}{v_1} = \frac{A_t^2}{v_2}$$

$$\frac{(A_i + A_r)(A_i - A_r)}{v_1} = \frac{A_t^2}{v_2}$$

$$\frac{(A_i + A_r)A_t}{v_1} = \frac{A_t^2}{v_2}$$

[Using equation (3), $(A_i - A_r) = A_t$]

$$(A_i + A_r) = \frac{v_1}{v_2} A_t \quad \text{... (4)}$$

By adding equations (3) and (4), we get,

$$2A_i = (1 + \frac{v_1}{v_2})A_t$$

$$A_t = \frac{2v_2}{v_1 + v_2} A_i$$

By subtracting equation (3) from equation (4), we get,

$$2A_r = (\frac{v_1}{v_2} - 1)A_t$$

$$2A_r = (\frac{v_1 - v_2}{v_2})(\frac{2v_2}{v_1 + v_2})A_i$$

[Substitute the value of A_t from equation (v)]

$$A_r = (\frac{v_1 - v_2}{v_1 + v_2})A_i \quad \text{... (4)}$$

Hence, if the wave is transitioning from the rarer to the denser medium, the amplitude of the reflected wave is determined by:

$$A_r = (\frac{v_1 - v_2}{v_1 + v_2})A_i$$

The amplitude of the transmitted wave becomes,

$$A_t = (\frac{2v_2}{v_1 + v_2})A_i$$

Note: If the wave travels from the denser to the rarer medium, then the amplitude of the reflected wave is given by:

$$A_r = (\frac{v_2 - v_1}{v_1 + v_2})A_i$$

The amplitude of the transmitted wave becomes,

$$A_t = (\frac{2v_2}{v_1 + v_2})A_i$$

Ex. A harmonic wave is travelling on string 1. At a junction with string 2, it is partly reflected and partly transmitted. The linear mass density of the second string is four times that of the first string, and the boundary between the two strings is at $x = 0$. If the expression for the incident wave is, $y_i = A_i \cos(k_1 x - \omega_1 t)$, then what are the expressions for the transmitted and reflected waves in terms of A_i, k_1, and ω_1? ?

Sol. Given that the linear mass density of the second string is four times that of the first string. It means that if the linear mass density of the first string is μ, then the linear mass density of the second string is 4μ.

Since the velocity of a wave in a medium is defined as $v = \sqrt{\frac{F}{\mu}}$ and the force acting on the strings is the same, then the velocity of the wave on string 2 becomes half of the velocity of the wave on string 1.

Thus, if the velocity of the wave on string 1 is v_1, then the velocity of the wave on string 2 becomes, $v_2 = \frac{v_1}{2}$

Therefore, the amplitude of the transmitted wave becomes.

$$A_t = (\frac{2v_2}{v_1 + v_2})A_i$$

$$A_t = (\frac{2(\frac{v_1}{2})}{v_1 + (\frac{v_1}{2})})A_i$$

$$A_t = \frac{v_1}{(\frac{3v_1}{2})} A_i$$
$$A_t = \frac{2}{3} A_i$$

Since the wave is travelling from the rarer to the denser medium, the amplitude of the reflected wave becomes.

$$A_r = (\frac{v_1 - v_2}{v_1 + v_2}) A_i$$
$$A_r = (\frac{v_1 - \frac{v_1}{2}}{v_1 + \frac{v_1}{2}}) A_i$$
$$A_r = \frac{(\frac{v_1}{2})}{(\frac{3v_1}{2})} A_i$$
$$A_r = \frac{1}{3} A_i$$

As the reflected wave is inverted compared to the incident wave, they are out of phase, or alternatively, we can state that the phase difference between the incident and reflected waves is π. Therefore, the expression of the reflected wave is,

$$y_r = \frac{1}{3} A_i \cos(k_1 x - \omega_1 t + \pi)$$

The transmitted wave is in phase with the incident wave and there is no phase difference between them. Thus, the velocity of the transmitted wave is, $v_2 = \frac{v_1}{2}$. If the angular frequency and the angular wave number of the transmitted wave is ω_2 and k_2, then,

$$v_2 = \frac{\omega_2}{k_2}$$
$$\frac{v_1}{2} = \frac{\omega_2}{k_2}$$
$$\frac{1}{2}(\frac{\omega_1}{k_1}) = \frac{\omega_2}{k_2}$$

Since frequency is independent of the medium $\omega_1 = \omega_2$

Therefore, $k_2 = 2k_1$

Hence, the equation of the transmitted wave is,

$$y_t = A_t \cos(k_2 x - \omega_2 t)$$
$$y_t = \frac{2}{3} A_i \cos(2k_1 x - \omega_1 t)$$