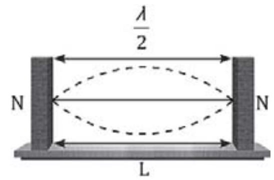
NORMAL MODES OF VIBRATION IN A STRING

Modes of Vibration of String fixed at Both Ends

1. Fundamental mode or First harmonic



By putting n = 1 in equations (3) and (4), we get,

$$L = \frac{\lambda}{2}$$
$$f = \frac{v}{2L}$$

Since $L = \frac{\lambda}{2}$ There is only one loop in between the nodes, as shown in the figure.

Now, considering v as the velocity of the two parent traveling waves, it can be expressed as follows:

$$v = \sqrt{\frac{F}{\mu}}$$

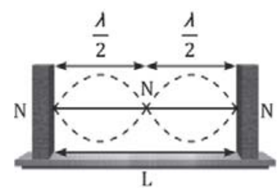
Hence, the expression for frequency becomes,

$$f = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$$

Thus, the minimum frequency required to generate the standing wave along the string fixed at both ends, also known as the fundamental frequency, is given by:

$$f_0 = \frac{v}{2L} = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$$

2. Second harmonic or first overtone



By putting n = 2 in equations (3) and (4), we get,

$$L = 2\left(\frac{\lambda}{2}\right)$$
$$f = 2\left(\frac{v}{2L}\right)$$

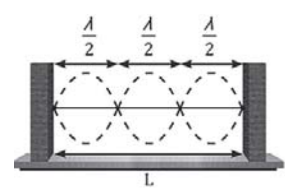
Since $L = 2\left(\frac{\lambda}{2}\right)$ In this case, there are two loops between the fixed ends, resulting in one node between them, as illustrated in the figure.

The frequency for this case is represented as follows:

$$f_1 = 2\left(\frac{v}{2L}\right) = 2\left[\frac{1}{2L}\sqrt{\frac{F}{\mu}}\right]$$

Note: Typically, the subscript of f denotes its overtone, while the coefficient preceding it represents $\frac{v}{2L}$ or $\frac{1}{2L}\sqrt{\frac{F}{\mu}}$ determines its harmonics.

3. Third harmonic or second overtone



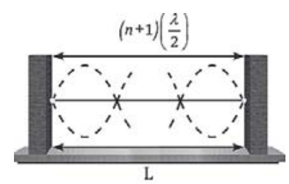
By putting n = 3 in equations (3) and (4), we get

$$L = 3\left(\frac{\lambda}{2}\right)$$
$$f = 3\left(\frac{v}{2L}\right)$$

Consequently, there are three loops between the fixed ends, indicating two nodes between them, as depicted in the figure. The frequency corresponding to this scenario is denoted as follows:

$$f_2 = 3\left(\frac{v}{2L}\right) = 3\left[\frac{1}{2L}\sqrt{\frac{F}{\mu}}\right]$$

4. $(\mathbf{n+1})^{\text{th}}$ harmonic or $\mathbf{n}^{\text{th}}$ overtone



Here, $L = (n+1)\left(\frac{\lambda}{2}\right)$, And hence, there are (n + 1) loops in between the fixed ends. This implies that there are n nodes in between the fixed ends. The frequency of this case is as follows:

$$f_n = (n+1)\left(\frac{v}{2L}\right) = (n+1)\left[\frac{1}{2L}\sqrt{\frac{F}{\mu}}\right]$$

Ex. The equation for the vibration of a string fixed at both the ends vibrating in its third harmonic is given by, $y = (0.4\ \text{cm}) \sin[(0.314\ \text{cm}^{-1})\, x] \cos[(600\pi\ \text{s}^{-1})\, t]$

(a) What is the frequency of the vibration?

(b) What are the positions of the nodes?

(c) What is the length of the string?

(d) What is the wavelength and the speed of two travelling waves that can interfere to give this vibration?

Sol. Given,

$y = (0.4\ cm)\sin[(0.314\ cm^{-1})x]\cos[(600\pi\ s^{-1})t]$

By comparing this with the general equation of the standing wave, i.e., $y = A\sin(kx)\cos(\omega t)$, we get, Amplitude, $A = 0.4$ cm

Angular wave number, $k = 0.314\ cm^{-1} = \frac{\pi}{10}\ cm^{-1}$

Angular frequency, $\omega = 600\pi\ s^{-1}$ (Here, π is expressed in rad, and hence, its unit is written as s^{-1}.)

(a) Angular frequency, $\omega = 600\pi\ s^{-1}$

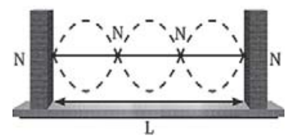
If f is the frequency of vibration, then we get,

$$\omega = 2\pi f$$
$$f = \frac{\omega}{2\pi}$$

By substituting the value of ω, we get,

$$f = \frac{600\pi}{2\pi} = 300\ Hz$$

(b) Since the string is vibrating in its third harmonic, there are three loops and two nodes in between the two fixed ends, which are also nodes.



$$\text{Since } k = \frac{\pi}{10}\ cm^{-1},$$
$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{(\frac{\pi}{10})} = 20\ cm$$

We know that the distance between each node is $\frac{\lambda}{2}$ there are four nodes and the position of the nodes are at 0 cm, 10 cm, 20 cm, and 30 cm.

(c) From the analysis given in part (b), we came to know that the two nodes at the fixed ends of the string are at 0 cm and 30 cm. Therefore, the length of the string is 30 cm.

(d) We know the following quantities: Angular wave number, $k = 0.314 cm^{-1} = \frac{\pi}{10} cm^{-1}$

Angular frequency, $\omega = 600\pi\ s^{-1}$ From the analysis done in part (a), we get, Frequency of vibration, f = 300 Hz From the analysis done in part (b), we get, Wavelength of the parent travelling waves, λ = 20 cm

Therefore, the velocity of the parent travelling waves that can interfere to give the given vibration is given by,

$$v = f\lambda = (300\ Hz) \times (20\ cm) = 6{,}000\ cm\ s^{-1} = 60\ ms^{-1}$$

Ex. Two wires are kept tight between the same pair of supports. The tensions in the wires are in the ratio 2 : 1, the radii are in the ratio 3 : 1, and the densities are in the ratio 1 : 2. Find the ratio of their fundamental frequencies.

Sol. Since two wires are kept between the same pair of supports, their length should be equal. Let the tension in the wires be T, the radius of the wires be R, and the density of the wires be ρ.

Given,
$$T_1 : T_2 = 2 : 1$$
$$R_1 : R_2 = 3 : 1$$
$$\rho_1 : \rho_2 = 1 : 2$$

The fundamental frequency of string 1 is given by,
$$(f_0)_1 = \frac{v_1}{2L}$$

The fundamental frequency of string 2 is given by,
$$(f_0)_2 = \frac{v_2}{2L}$$

Therefore,
$$\frac{(f_0)_1}{(f_0)_2} = \frac{v_1}{v_2}$$
$$\frac{(f_0)_1}{(f_0)_2} = \sqrt{\frac{T_1}{\mu_1}\frac{\mu_2}{T_2}}$$
$$\frac{(f_0)_1}{(f_0)_2} = \sqrt{\frac{T_1}{T_2}\frac{\rho_2(\pi R_2^2)}{\rho_1(\pi R_1^2)}}\ (\text{As } \mu = \rho A = \rho \times \pi R^2)$$
$$\frac{(f_0)_1}{(f_0)_2} = \sqrt{\frac{T_1}{T_2}\frac{\rho_2}{\rho_1}(\frac{R_2}{R_1})^2}$$
$$\frac{(f_0)_1}{(f_0)_2} = \sqrt{\frac{2}{1} \times \frac{2}{1} \times (\frac{1}{3})^2}$$
$$\frac{(f_0)_1}{(f_0)_2} = \frac{2}{3}$$

Hence, the ratio of their fundamental frequencies is 2: 3.

Ex. Three resonant frequencies of a string are 90, 150, and 210 Hz.

(a) Find the highest possible fundamental frequency of vibration of this string.

(b) Which harmonics of the fundamental are the given frequencies?

(c) Which overtones are these frequencies?

(d) If the length of the string is 80 cm, what is the speed of a transverse wave on this string?

Sol. **(a)** Given, the three resonant frequencies of a string are 90, 150, and 210 Hz. The highest number by which all the three given numbers are divisible, i.e., HCF of these three numbers is 30. Therefore, the highest possible fundamental frequency of vibration of this string is given by, $f_0 = 30$ Hz.

(b) The frequency of 90 Hz can be written as follows:

$$3 \times 30 \text{ Hz} = 3f_0$$

Therefore, this is the third harmonic of the fundamental frequency.

Similarly, the frequency of 150 Hz can be written as follows:

$$5 \times 30 \text{ Hz} = 5f_0$$

Similarly, the frequency of 210 Hz can be written as follows:

$$7 \times 30 \text{ Hz} = 7f_0$$

Hence, the given resonant frequencies are third, fifth, and seventh harmonics of the fundamental frequency.

(c) Since we know that $(n + 1)^{th}$ harmonic is equivalent to nth overtone, the given resonant frequencies are second, fourth, and sixth overtones of the fundamental frequency.

(d) Fundamental frequency, f 0 = 30 Hz

Given, length of the string, L = 80 cm = 0.8 m If v is the velocity of the transverse wave travelling on the string, then we get,

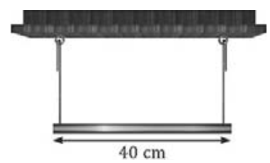
$$f_0 = \frac{v}{2L}$$

$$v = f_0 \times 2L$$

$$v = (30 \text{ Hz}) \times (2 \times 0.8 \text{ m}$$

$$v = 48 \text{ ms}^{-1}$$

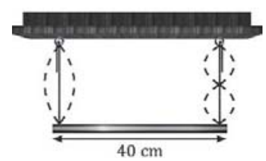
Ex. A uniform horizontal rod of length 40 cm and mass 1.2 kg is supported by two identical wires, as shown in the figure. Where should a mass of 4.8 kg be placed on the rod so that the same tuning fork may excite the wire on left into its fundamental vibrations and that on right into its first overtone? (Take g = 10 ms^{-2})



Sol. Given, Length of the rod, L = 40 cm

Mass of the rod, m = 1.2 kg

Mass to be placed on the rod is, m = 4.8 kg



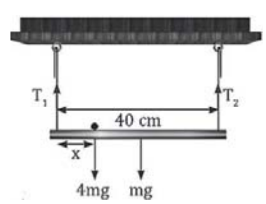
The left wire vibrates with fundamental frequency (first harmonic, f_1), and the right wire vibrates with the frequency of first overtone (second harmonic, f_2), as shown in the figure.

Suppose the particle is put at a distance x from the left wire, as shown in the figure. Let the tension in the left wire be T_1 and that in the right wire be T_2. Hence, by balancing the forces, we get,

$$T_1 + T_2 = 4mg + mg$$

$$T_1 + T_2 = 5mg \quad \text{... (1)}$$

Since both the wires are excited by the same source, the frequency of vibration of both the wires are the same. Therefore,



$$f_1 = f_2$$

$$1\left(\frac{v_1}{2L_1}\right) = 2\left(\frac{v_2}{2L_2}\right)$$

$$v_1 = 2v_2$$

[Since the length of both the rods are equal]

$$\sqrt{\frac{T_1}{\mu_1}} = 2\sqrt{\frac{T_2}{\mu_2}}$$

$$\sqrt{T_1} = 2\sqrt{T_2}$$

[Since both the strings are the same, their linear mass density is also the same]

$$T_1 = 4T_2 \quad \text{... (1)}$$

By putting equation (2) in equation (1), We get, $5T_2 = 5mg$

$T_2 = mg$

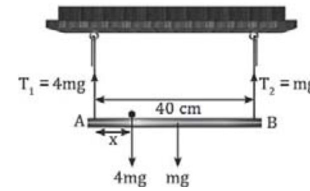
Therefore, from equation (2),

We get, $T_1 = 4$ mg

Since the rod is in angular equilibrium, the net torque about any point is zero.

Let us calculate the torque about point A.

Therefore, about point A,



1. The torque of T_1 is zero.
2. The torque of 4 mg is (4mg) x in the clock wise direction.
3. The torque of mg (mg) $\frac{L}{2}$ in the clock wise direction.
4. The torque of T_2 is $T_2 L$ in the antilock wise direction.

By balancing the torque about point A, we get,

$$\&4mgx + mg\frac{L}{2} = T_2 L$$

$$4mgx + mg\frac{L}{2} = mgL$$

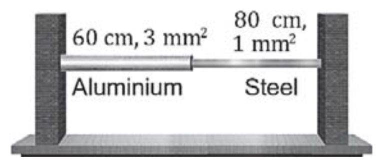
$$4x + \frac{L}{2} = L$$

$$x = \frac{L}{8}$$

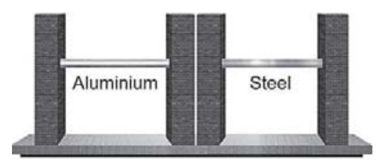
$$x = \frac{40}{8} = 5 \text{ cm}$$

Therefore, the mass of 4.8 Kg should be placed at 5 cm away from the left wire.

Ex. An aluminimum wire of length 60 cm is joined to a steel wire of length 80 cm and is stretched between two fixed supports. The tension produced is 40 N. The cross-sectional area of the steel wire is 1 mm^2 and that of the aluminimum wire is 3 mm^2. What could be the minimum frequency of a tuning fork that can produce standing waves in the system with the joint as a node? The density of aluminimum wire is 2.6 g cm^{-3} and that of steel wire is 7.8 g cm^{-3}.



Sol. Since we require a standing wave with the joint of the two wires as a node, we can treat both the wires as fixed at both the ends as shown in the figure. Now, the same source (the tuning fork) produces the waves on both the wires. Therefore, the frequency of the waves is the same for the wires, but it does not necessarily mean that they are in the same harmonics of the fundamental mode Suppose that the aluminimum wire is oscillating in its n_a^{th} harmonic, whereas the steel wire is oscillating in its n_a^{th} harmonic. If we assume that the velocity of the parent waves producing standing waves in the aluminimum and steel wires are v_a and v_s , respectively, and the length of the aluminimum and steel wires are l a and l s respectively, then we have,



$$n_a\left(\frac{v_a}{2l_a}\right) = n_s\left(\frac{v_s}{2l_s}\right)$$

$$\frac{n_a}{n_s} = \frac{l_a}{l_s}\frac{v_s}{v_a}$$

$$\frac{n_a}{n_s} = \frac{l_a}{l_s}\sqrt{\frac{F_s}{\mu_s}\frac{\mu_a}{F_a}}$$

Since the tension on both the strings is the same, $F_s = F_a$. we know that the linear mass density, $\mu = \rho S$ where ρ is the density and is the cross-sectional area.

$$\frac{n_a}{n_s} = \frac{l_a}{l_s}\sqrt{\frac{\mu_a}{\mu_s}}$$

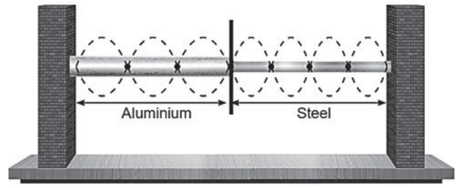
$$\frac{n_a}{n_s} = \frac{l_a}{l_s}\sqrt{\frac{\rho_a S_a}{\rho_s S_s}}$$

$$\frac{n_a}{n_s} = \frac{60}{80}\sqrt{\frac{2.6\times3}{7.8\times1}}$$

$$\frac{n_a}{n_s} = \frac{60}{80}\sqrt{\frac{1\times3}{3\times1}}$$

$$\frac{n_a}{n_s} = \frac{3}{4}$$

Therefore, we come to the conclusion that if one source produces standing waves with the same frequency on both the strings together so that the joint acts as a node, then the harmonics of the waves on the aluminimum and steel wires will be in the ratio of 3 : 4. However, for the minimum frequency, n_a and n_s should be equal to 3 and 4, respectively. Hence, there are three loops in the aluminimum wire and four loops in the steel wire.



Now, it is given that,
Length of the aluminimum wire, $l_a = 60$ cm $= 0.6$ m
Length of the steel wire, $l_s = 80$ cm $= 0.8$ m
Tension in the wires, $F_a = F_s = 40$ N
Cross-sectional area of the aluminimum wire, $S_a = 3\ mm^2 = 3 \times 10^{-6}\ m^2$
Cross-sectional area of the steel wire, $S_s = 1\ mm^2 = 1 \times 10^{-6}\ m^2$
Density of aluminum, $\rho_a = 2.6\ g\ cm^{-3} = 2.6 \times 10^3\ kg\ m^{-3}$
Density of steel, $\rho_s = 7.8\ g\ cm^{-3} = 7.8 \times 10^3\ kg\ m^{-3}$
Therefore, the minimum frequency is given by,

$$f_{min} = n_a\left(\frac{v_a}{2l_a}\right) = n_s\left(\frac{v_s}{2l_s}\right)$$

$$f_{min} = \left(\frac{n_a}{2l_a}\right)\sqrt{\frac{F_a}{\mu_a}} = \left(\frac{n_a}{2l_a}\right)\sqrt{\frac{F_a}{\rho_a S_a}}$$

By substituting the required values for the aluminimum wire, we get,

$$f_{min} = \left(\frac{3}{2\times0.6}\right)\sqrt{\frac{40}{(2.6\times10^3)(3\times10^{-6})}}$$

$$f_{min} = \left(\frac{1}{0.4}\right)\sqrt{\frac{40}{(7.8\times10^{-3})}}$$

$$f_{min} = 2.5 \times \sqrt{5128.21}$$

$$f_{min} \approx 180\ \text{Hz}$$

Therefore, the minimum frequency of a tuning fork that can produce standing waves in the system with the joint as a node is 180 Hz.

Ex. A string is stretched by a block going over a pulley. The string vibrates in its tenth harmonic in unison with a particular tuning fork. When a beaker containing water is brought under the block, so that the block is completely immersed in the water, the string vibrates in its eleventh harmonic. Find the density of the material of the block.

Sol. Consider the mass of the block as m. Therefore, initially, the tension in the string,

$$T = mg \quad \dots (1)$$

When a beaker containing water is brought under the block so that the block is completely dipped in the beaker, due to the buoyant force on the block, the apparent weight of the block and the tension in the string changes. Hence, the harmonic vibration of the string also changes. Assume that the tension in the string for this case becomes T' and the buoyant force on the block is B.

Then,

$$T' + B = mg$$

$$T' = mg - B \quad \dots (2)$$

Since the source is same for the whole process, the frequency remains constant. However, the harmonic vibration of the string before and after the block is dipped into water changes because of the difference in tension of the string due to the placement of the beaker. Initially, the string vibrates in its tenth harmonic and finally, it vibrates in its eleventh harmonic. If the length of the string is L and the linear mass density is μ, then,

$$\frac{10}{2L}\sqrt{\frac{T}{\mu}} = \frac{11}{2L}\sqrt{\frac{T'}{\mu}}$$

$$\frac{10}{11} = \sqrt{\frac{T'}{T}}$$

$$\frac{100}{121} = \frac{T'}{T}$$

&By substituting the values of T′ and T from equation (1) and (2), we get,

$$\frac{mg-B}{mg} = \frac{100}{121}$$

&If the density of the block and the water are ρ_s and ρ_w, then,

$\frac{\rho_s Vg-\rho_w Vg}{\rho_s Vg} = \frac{100}{121}$ [Where V is the volume of the block]

$$\frac{\rho_s-\rho_w}{\rho_s} = \frac{100}{121}$$
$$121\rho_s - 121\rho_w = 100\rho_s$$
$$21\rho_s = 121\rho_w$$

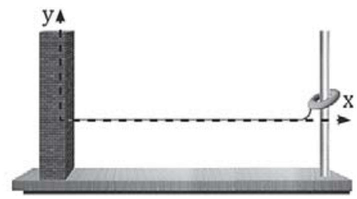
$\rho_s = \frac{121}{21}$ g cm^{-3}[Density of the water is, $\rho_w = 1$ g cm^{-3}]

$$\rho_s = 5.76 \text{ g cm}^{-3}$$
$$\rho_s \approx 5.8 \times 10^3 \text{ kg m}^{-3}$$

Therefore, the density of the material of the block is 5.8×10^3 kg m^{-3}.

String Fixed at One End

Take into account a string with a fixed end and a free end, both ends having a distance of L between them. In order for a standing wave to be perfect on the string, a point of no displacement, known as a node, will be present at the fixed end (x = 0), while a point of maximum displacement, known as an antinode, will be present at the free end (x = L).



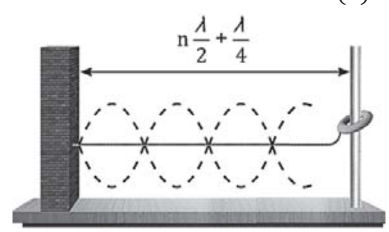
Assume that the equation of the standing wave is,

$$y = 2A\sin(kx)\cos(\omega t)$$

For the antinode at x = L, y is $\pm 2A$ for all time t. Therefore,

$$\sin(kL) = \pm 1$$
$$kL = (2n \pm 1)\frac{\pi}{2}$$
$$\left(\frac{2\pi}{\lambda}\right)L = (2n \pm 1)\frac{\pi}{2}$$
$$L = (2n \pm 1)\frac{\lambda}{4}$$
$$L = \left(\frac{n\lambda}{2} \pm \frac{\lambda}{4}\right) \quad \text{... (1)}$$

Therefore, the shape of the standing wave in the string, anchored at one end, consists of n complete loops and an additional half loop, as depicted in the illustration. Hence, for achieving a flawless standing wave on a string anchored at one end, the length of the string needs to be a multiple of an odd number. $\frac{\lambda}{4}$



If v and f are the velocity and the frequency, respectively, of those two parent waves whose superposition gives the standing wave, then we can rewrite equation (1) as follows:

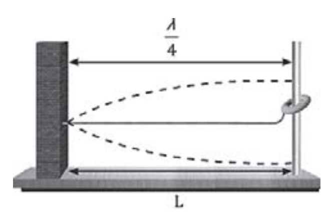
$$L = (2n \pm 1)\frac{\lambda}{4}$$
$$L = (2n \pm 1)\frac{v}{4f}$$
$$f = (2n \pm 1)\frac{v}{4L} \quad \text{... (2)}$$

Modes of Vibration of a String Fixed at One End

1. Fundamental mode or first harmonic

By substituting n = 0 in equations (1) and (2), we get,

$$L = \frac{\lambda}{4} \text{ and } f = \frac{V}{4L}$$



Since $L = \frac{\lambda}{4}$ there will be a half loop in between the nodes and antinodes as shown in the figure. Therefore, the fundamental frequency is given by,

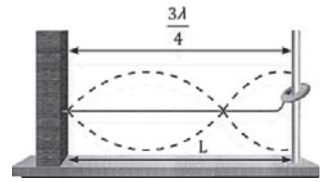
$$f_0 = \frac{v}{4L} = \frac{1}{4L}\sqrt{\frac{F}{\mu}}$$

2. First overtone or third harmonic

By substituting n = 1 in equations (1) and (2), we get,

$$L = \frac{\lambda}{2} + \frac{\lambda}{4} = 3(\frac{\lambda}{4}) \text{ and } f = 3(\frac{v}{4L})$$

Since $L = 3(\frac{\lambda}{4})$, there will be one and a half loop in between the fixed end and the free end as shown in the figure. The frequency for this case is represented by,



$$f_1 = 3(\frac{v}{4L}) = 3[\frac{1}{4L}\sqrt{\frac{F}{\mu}}] = 3f_0$$

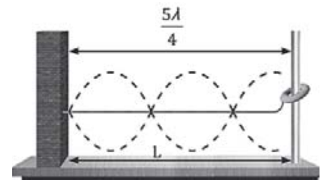
3. Second overtone or fifth harmonic

By substituting n = 2 in equations (1) and (2), we get,

$$L = \lambda + \frac{\lambda}{4} = 5(\frac{\lambda}{4}) \text{ and } f = 5(\frac{v}{4L})$$

Since $L = 5(\frac{\lambda}{4})$, there will be two and a half loops in between the fixed and free ends as shown in the figure.

The frequency for this case is represented by,

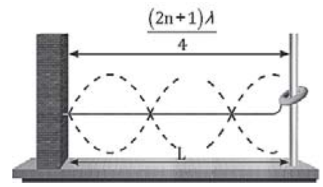


$$f_2 = 5(\frac{v}{4L}) = 5[\frac{1}{4L}\sqrt{\frac{F}{\mu}}] = 5f_0$$

4. n^{th} overtone or $(2n + 1)^{th}$ harmonic

Here, $L = (2n + 1)(\frac{\lambda}{4})$ and hence, there will be n and one half loop in between the fixed and free ends. The frequency of this case is,

$$f_n = (2n + 1)(\frac{v}{4L}) = (2n + 1)[\frac{1}{4L}\sqrt{\frac{T}{\mu}}] = (2n + 1)f_0$$



Ex. A 2 m long rope having a mass of 80 g is fixed at one end and is tied to a light string at the other end. The tension in the string is 256 N.

(a) Find the frequencies of the fundamental and the first two overtones.

(b) Find the wavelength in the fundamental and the first two overtones.

Sol. Given,

The length of the long rope is, L = 2 m

The mass of the long rope is, m = 80 g = 0.08 kg

The tension in the string is, F = 256 N

Therefore, the mass per unit length of the rope is,

$$\mu = \frac{m}{L} = \frac{0.08}{2} = 0.04 \text{kgm}^{-1}$$

(a) The fundamental frequency is

$$f_0 = \frac{1}{4L}\sqrt{\frac{F}{\mu}}$$

$$f_0 = \frac{1}{4\times 2}\sqrt{\frac{256}{0.04}}$$

$$f_0 = \frac{1}{8}\times\frac{16}{0.2}$$

$$f_0 = 10 \text{ Hz}$$

Hence, the fundamental frequency is, $f_0 = 10$ Hz. Therefore, the frequency of the first overtone is $f_1 = 3f_0 = 30$ Hz and the frequency of the second overtone is $f_2 = 5f_0 = 50$ Hz.

(b) For fundamental frequency, the wavelength of the wave is related to the length of the rope as,

$$L = \frac{\lambda}{4}$$

$$\lambda = 4L$$

$$\lambda = 4 \times 2 = 8 \text{ m}$$

Therefore, the wavelength of the fundamental mode is, $\lambda_0 = 8$ m. For the wavelength of the first overtone,

$$L = \frac{3\lambda}{4}$$

$$\lambda = \frac{4L}{3}$$

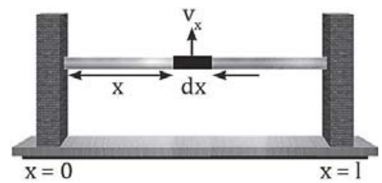
$$\lambda = \frac{\lambda_0}{3}$$

$$\lambda = \frac{8}{3} = 2.667\text{m}$$

Therefore, the wavelength of the first overtone is, $\lambda_1 = 2.667$ m. Following the same procedure, we get the wavelength of the second overtone as,

$$\lambda_2 = \frac{\lambda_0}{5} = \frac{8}{5} = 1.6 \text{ m}$$

Ex. A string tied between x = 0 and x = l vibrates in the fundamental mode. It has an amplitude A, tension T, and mass per unit length μ. Find the total energy of the string.



Sol. Let the equation of the standing wave be, y = A sin (kx) cos (ωt) Given that the string is fixed at both the ends and vibrating in its fundamental mode, the length of the string is $l = \frac{\lambda}{2}$, where λ is the wavelength of the wave.

Thus, the angular wave number is, $k = \frac{2\pi}{\lambda} = \frac{2\pi}{2l} = \frac{\pi}{l}$

We know that the velocity of the parent travelling wave is,

$$v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{T}{\mu}}$$

Consider a small element of length dx at a distance x from the left end and the mass of the element as dm = μdx

If v_x is the velocity of that element, then the kinetic energy associated with that small element is,

$$dK = \frac{1}{2}(dm)(v_x)^2$$

$$dK = \frac{1}{2}(\mu dx)(\frac{\partial y}{\partial t})^2$$

$$dK = \frac{1}{2}(\mu dx)[A(-\omega)\sin(kx)\sin(\omega t)]^2$$

$$dK = \frac{\mu\omega^2A^2}{2}\sin^2(kx)\sin^2(\omega t)dx$$

Therefore, the net kinetic energy for the whole string at any time is,

$$K(t) = \frac{\mu\omega^2A^2}{2}\sin^2(\omega t)\int_0^l \sin^2(kx)dx$$

Now by solving the equation separately, we get the following.

$$\int_0^l \sin^2(kx)dx = \frac{1}{2}\int_0^l 2\sin^2(kx)\,dx$$

$$\int_0^l \sin^2(kx)dx = \frac{1}{2}\int_0^l[1-\cos(2kx)]\,dx$$

$$\int_0^l \sin^2(kx)dx = \frac{1}{2}[x - \frac{1}{2k}\sin(2kx)]_0^l$$

$$\int_0^l \sin^2(kx)dx = \frac{1}{2}[x - \frac{l}{2\pi}\sin(\frac{2\pi x}{l})]_0^l$$

$$\int_0^l \sin^2(kx)dx = \frac{1}{2}[[l-0] - \frac{l}{2\pi}[\sin(2\pi) - \sin(0)]]$$

$$\int_0^l \sin^2(kx)dx = \frac{l}{2}$$

By substituting the value obtained from the integration in the kinetic energy expression, we get,

$$K(t) = \frac{\mu\omega^2A^2l}{4}\sin^2(\omega t)$$

Therefore, the maximum value of kinetic energy is,

$$K_{max} = \frac{\mu\omega^2A^2l}{4}$$

Since for SHM,

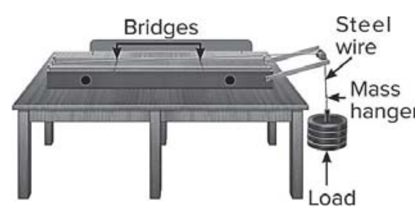
Maximum kinetic energy = Total mechanical energy = Maximum potential energy

The travelling waves that produce the standing wave executes SHM and hence, the total mechanical energy is the maximum kinetic energy.

Therefore, the total energy is, $E = \frac{\mu\omega^2A^2l}{4}$

Sonometer

A sonometer is made up of the following:



1. A wooden enclosure, alternatively referred to as a sound box.
2. A string is fastened at one end of the enclosure, while a weight is suspended from the other end.

Within the enclosure, there are two adjustable bridges that establish the nodes. The spacing between these bridges can be altered to modify the length of the string that vibrates.
In sonometer experiments, we have the flexibility to alter either the mass or the vibrating length of the wire. The diagram illustrates the construction of a sonometer.

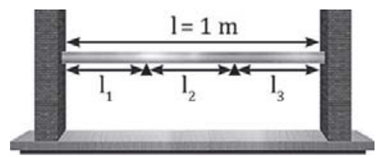
(a) Law of length:
The fundamental frequency of vibration of a string anchored at both ends varies inversely with the length of the string, provided that the tension and linear mass density remain constant. This relationship can be expressed mathematically as:
$f \propto \frac{1}{L} \Rightarrow \frac{f_1}{f_2} = \frac{L_2}{L_1}$ (If the tension F and linear mass density μ are constants)

(b) Law of tension:
The fundamental frequency of vibration of a string anchored at both ends is directly proportional to the square root of its tension, assuming the length and linear mass density of the string remain constant. This relationship can be expressed mathematically as:
$f \propto \sqrt{T} \Rightarrow \frac{f_1}{f_2} = \sqrt{\frac{T_1}{T_2}}$ (If the length L and linear mass density μ are constants)

(c) Law of mass:
The fundamental frequency of vibration of a string anchored at both ends varies inversely with the square root of its mass, provided that the length and tension in the string remain constant. Mathematically, it can be expressed as:
$f \propto \frac{1}{\sqrt{\mu}} \Rightarrow \frac{f_1}{f_2} = \sqrt{\frac{\mu_2}{\mu_1}}$ (If the length L and the tension T are constants)

Ex. For a sonometer wire with a length of 1 meter between fixed ends, where should the two bridges be positioned underneath the wire so that the fundamental frequencies of the three segments of the wire are in the ratio 1:2:3?

Sol. Given,



The length of the string fixed at both ends is, l = 1 m. suppose that the two bridges are placed in such a way that the string is divided into three parts of length l_1, l_2, and l_3. The three parts of the strings behave as three individual strings fixed at both the ends.
It is also given that the parts of the wire have their fundamental frequencies in the ratio 1: 2: 3. Since the tension in the three parts of the string is equal and the linear mass density for each part is also equal, the fundamental frequencies related to the three parts are.

$$f_1 = \frac{v}{2l_1}, f_2 = \frac{v}{2l_2}, \text{and } f_3 = \frac{v}{2l_3}$$

Therefore, the ratio of the fundamental frequency of each part of the string is,

$$f_1 : f_2 : f_3 = 1 : 2 : 3$$
$$\frac{v}{2l_1} : \frac{v}{2l_2} : \frac{v}{2l_3} = 1 : 2 : 3$$
$$\frac{1}{l_1} : \frac{1}{l_2} : \frac{1}{l_3} = 1 : 2 : 3$$
$$l_1 : l_2 : l_3 = 1 : \frac{1}{2} : \frac{1}{3}$$
$$l_1 : l_2 : l_3 = 6 : 3 : 2 \text{ (Multiplying the ratio on RHS by 6)}$$

Therefore, $l_1 = \frac{6}{11} l = \frac{6}{11} \text{m}$ (l = Total length of the string = 1 m)

$$l_2 = \frac{3}{11} l = \frac{3}{11} \text{m}$$
$$\& l_3 = \frac{2}{11} l = \frac{2}{11} \text{m}$$

Therefore one bridge is $\frac{6}{11}$ m away from the left end and the other bridge is $\frac{2}{11}$ m away from the right end.