ENERGY CALCULATION IN WAVES

Speed of a Transverse Pulse on a String

The velocity of a wave traveling along a string is contingent upon both the tension (T) and the linear mass density (μ) of the string.

$$\text{Speed} \propto \text{Tension of the vibrating source (T)}$$

$$\text{Speed} \propto \frac{1}{\text{Mass per unit length }(\mu)}$$

Therefore, the speed of wave is, $v = \sqrt{\frac{T}{u}} = \frac{\text{Elastic property}}{\text{Inertial property}}$

Ex. A string of length 5.5 m has a mass of 0.035 kg. If the tension in the string is 77 N, find the speed of a wave on the string.

(A) $77\ ms^{-1}$ (B) $102\ ms^{-1}$ (C) $110\ ms^{-1}$ (D) $164\ ms^{-1}$

Sol. We have, Tension of the string, T = 77 N

Length of the string, l = 5.5 m

Mass of the string, m = 0.035 kg

The speed of a wave propagating along the string is given as follows.

$$v = \sqrt{\frac{T}{\mu}}$$

Mass per unit length is, $\mu = \frac{m}{l} = \frac{0.035}{5.5}$

Substituting the value of μ, we get the following:

$$v = \sqrt{\frac{77 \times 5.5}{0.035}}$$

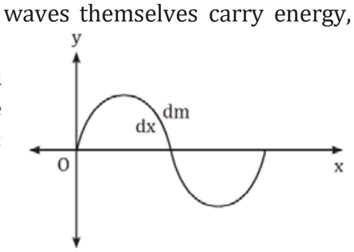
$$v = 110\ ms^{-1}$$

Thus, option (C) is the correct answer.

Energy Calculation in Waves

Kinetic energy

As we understand, as a wave travels through a medium, the particles comprising the medium undergo sinusoidal oscillations characterized by simplicity and harmonic nature. Each particle possesses energy due to these oscillations. Simultaneously, the waves themselves carry energy, facilitating energy transfer within the medium.



Let's examine a minute segment of length dx and mass dm within a string as a wave progresses through it, as depicted in the illustration. Consequently, the kinetic energy (KE) linked to this tiny mass element, dm, can be expressed as follows:

$$dK = \frac{1}{2} dm\left(v_p^2\right)$$

$$\text{Also, } v_p = \frac{\partial y}{\partial t} = \omega A\cos(\omega t - kx)$$

Mass of the small element, $dm = \mu dx$

Substituting the values of v_p and dm, we get the following:

$$dK = \frac{1}{2} \times \mu dx \times \omega^2 A^2 \cos^2(\omega t - kx)$$

Kinetic energy per unit length,

$$\frac{dK}{dx} = \frac{1}{2}\mu\omega^2 A^2 \cos^2(\omega t - kx)$$

Potential energy

Consider a string. We'll examine a specific segment, denoted as ds, of the string under three distinct circumstances, as illustrated in the figure.

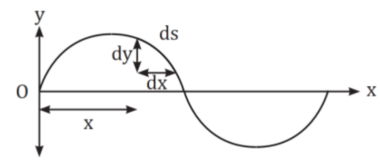
1. At the outset, the string remains unscratched, and no wave pulse is generated within it.

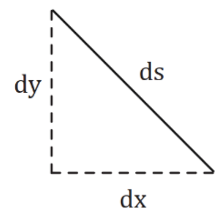
2. At t = 0, a wave pulse propagates through it. Subsequently, we observe that the element ds experiences an increase in its length.

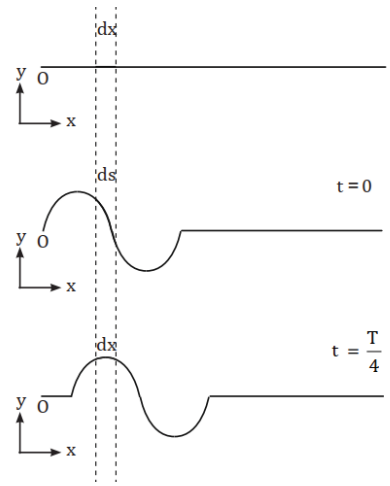
3. At time $t = \frac{T}{4}$ A wave pulse propagates within it, and we observe that the length of element ds remains identical to that in situation (1).

It is evident that at t = 0, when a pulse traverses the string, there is a minor extension in the length of element ds.

$$ds^2 = dx^2 + dy^2$$







The work done by the tension force, dU = F (ds – dx) (Since (ds – dx) is the displacement)
Here, the only force acting is tension (T). Therefore,

$$dU = T(ds - dx) = T\left(\sqrt{dx^2 + dy^2} - dx\right)$$

(Since $ds^2 = dx^2 + dy^2$)

$$dU = Tdx\left(\sqrt{1 + (\frac{dy}{dx})^2} - 1\right) \quad \text{... (1)}$$

Since dy and dx are very small, we can use the binomial expansion:

$$(1 + x)^n = 1 + nx$$

By applying the mentioned binomial expansion in equation (i), we get the following:

$$dU = Tdx(1 + \frac{1}{2}(\frac{dy}{dx})^2 - 1)$$

$$dU = \frac{Tdx}{2}(\frac{dy}{dx})^2$$

We know, $y = A\sin(\omega t - kx)$

$$\frac{dy}{dx} = -Ak\cos(\omega t - kx)$$

Also,

$$v = \sqrt{\frac{T}{\mu}}$$

$$T = v^2\mu$$

Substituting all the values, we get the following:

$$dU = v^2\mu dx \times \frac{1}{2} \times A^2k^2\cos^2(\omega t - kx)$$

The potential energy per unit length is given as follows:

$$\frac{dU}{dx} = \frac{1}{2}\mu\omega^2A^2\cos^2(\omega t - kx)$$

Upon comparing the equations for kinetic energy and potential energy, we note their similarity. This implies that kinetic energy and potential energy are in phase with each other, unlike in Simple Harmonic Motion (SHM) where potential energy decreases as kinetic energy increases, and vice versa.

Total Energy

Total mechanical energy = Kinetic energy + Potential energy

Potential energy at any instant, $dU = \frac{1}{2}\mu\omega^2A^2\cos^2(\omega t - kx)\,dx$

Kinetic energy at any instant, $dK = \frac{1}{2}\mu\omega^2A^2\cos^2(\omega t - kx)\,dx$

Total energy at any instant, $dE = dK + dU$

$$dE = \frac{1}{2}\mu\omega^2A^2\cos^2(\omega t - kx)\,dx + \frac{1}{2}\mu\omega^2A^2\cos^2(\omega t - kx)\,dx$$

$$dE = \mu\omega^2A^2\cos^2(\omega t - kx)\,dx$$

Average power transmitted,

$$\left\langle\frac{dE}{dt}\right\rangle = P_{avg} = \left\langle\frac{dK}{dt}\right\rangle + \left\langle\frac{dU}{dt}\right\rangle$$

$$\left\langle\frac{dE}{dt}\right\rangle = P_{avg} = \left\langle\frac{1}{2}\mu\omega^2A^2\cos^2(\omega t - kx)\frac{dx}{dt}\right\rangle + \left\langle\frac{1}{2}\mu\omega^2A^2\cos^2(\omega t - kx)\frac{dx}{dt}\right\rangle$$

Where, $\frac{dx}{dt} = v$ is the velocity of the wave.

$$P_{avg} = \frac{1}{4}\mu v\omega^2 A^2 + \frac{1}{4}\mu v\omega^2 A^2 \left(\text{Since average value of } \cos^2 x = \frac{1}{2}\right)$$

$$P_{avg} = \frac{1}{2}\mu v\omega^2 A^2$$

Substituting $\omega = 2\pi f$,

$$P_{avg} = 2\pi^2 f^2 \mu v A^2$$

Energy Density

Mathematically, the energy density can be found as follows:

$$\left\langle\frac{dE}{dx}\right\rangle = \left\langle\frac{dK}{dx}\right\rangle + \left\langle\frac{dU}{dx}\right\rangle$$

$$\left\langle\frac{dE}{dx}\right\rangle = \frac{1}{4}\mu\omega^2 A^2 + \frac{1}{4}\mu\omega^2 A^2$$

$$\left\langle\frac{dE}{dx}\right\rangle = 2\pi^2 f^2 \mu A^2$$

Wave Intensity

The intensity of a wave is described as the average energy flux within the medium per unit time per unit perpendicular area, or as the average power transmitted per unit area perpendicular to the direction of wave propagation.

$$P_{avg} = 2\pi^2 f^2 \mu v A^2$$

Also, $$\mu = \frac{m}{l}$$

$$P_{avg} = \frac{2\pi^2 f^2 m v A^2}{l}$$

Therefore, the wave intensity is given as follows:

$$I = \frac{P_{avg}}{\text{Area}} = \frac{2\pi^2 f^2 m v A^2}{l \times \text{Area}}$$

$$I = 2\pi^2 f^2 \rho v A^2$$

(Since $l \times$ Area $= V$ and $\frac{m}{V} = \rho$, density of the medium)

Therefore, $I \propto A^2$ and $I \propto f^2$

Ex. The amplitude of a wave is doubled and the frequency is reduced to one-fourth. What will be the intensity of the wave at the same point?

(A) Increased to double
(B) Increased to four times
(C) Decreased to half
(D) Decreased to one-fourt

Sol. Given,

$$A_2 = 2A_1$$

$$f_2 = \frac{f_1}{4}$$

Also, we know, $I \propto A^2$ and $I \propto f^2$

$$I \propto A^2 f^2$$

Therefore, $$\frac{I_1}{I_2} = \frac{A_1^2 f_1^2}{A_2^2 f_2^2}$$

$$\frac{I_1}{I_2} = \left(\frac{A_1}{A_2}\right)^2 \left(\frac{f_1}{f_2}\right)^2$$

$$\frac{I_1}{I_2} = \left(\frac{A_1}{2A_1}\right)^2 \left(\frac{4f_2}{f_2}\right)^2$$

$$I_1 = 4I_2$$

$$I_2 = \frac{I_1}{4}$$

Thus, option (D) is the correct answer.