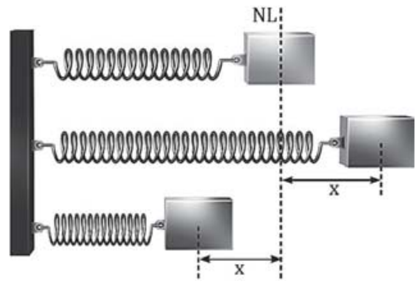
SPRING BLOCK SYSTEM IN SHM

Spring-Mass System



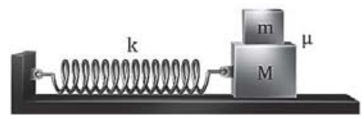
Consider a mass m connected to a spring at its natural length, meaning the spring is neither stretched nor compressed.

Assuming k is the spring constant, and the spring is either stretched or compressed by a distance x from its natural length, the restoring force of the spring is given by $F = -kx$. This force is the driving factor behind the occurrence of Simple Harmonic Motion (SHM).

Consequently, the spring-mass system will undergo simple harmonic motion, and the oscillation's time period will be.

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Ex. In the given figure, the horizontal plane is smooth. The friction coefficient between the two blocks is μ which is sufficient to prevent the upper block from slipping.



(a) If the system is slightly displaced and released, find the time period.

(b) Find the magnitude of the frictional force between the blocks when the displacement from the mean position is x.

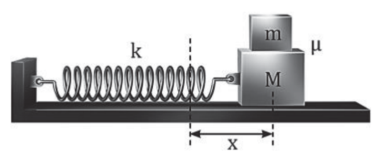
(c) What can be the maximum amplitude if the upper block does not slip relative to the lower block?

Sol. **(a)** Since the friction coefficient is sufficient to prevent the upper block from slipping, they will move together. So, the two blocks together can be considered as the system. Hence, the mass of the system becomes $(M + m)$ and they will execute SHM together as soon as the system is slightly displaced and released.

Therefore, the time period of SHM is $T = 2\pi\sqrt{\frac{M+m}{k}}$

From this, it can also be said that the angular frequency of SHM is $\omega = \sqrt{\frac{k}{M+m}}$.

(b) The system is displaced by x from the mean position, i.e., from the natural length of the spring. Hence, the system will be accelerated towards the mean position due to the restoring force of the spring and the magnitude of the acceleration will be $a = \omega^2 x$

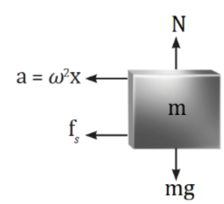


Now, if we consider the block of mass m to be our concerned system, then the forces on it will be:

1. Gravitational force (mg) in the downward direction

2. Normal reaction force (N) in the upward direction

3. Since the spring is attached to the block of mass M, the acceleration of the block of mass m is produced by the static frictional force, f s . The free body diagram of the upper block is shown in the figure.



Therefore, the magnitude of the frictional force between the blocks is,

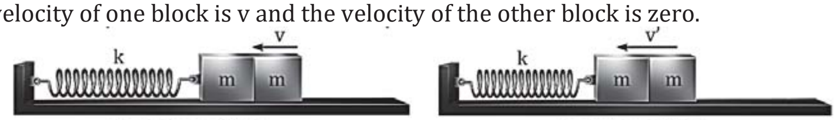
$$f_s = m|a|$$
$$f_s = m\omega^2 x$$
$$f_s = m\left(\frac{k}{M+m}\right)x$$

(c) A point up to which the static frictional force, f_s, between the blocks prevents their relative slipping defines the maximum amplitude of the SHM. The maximum value of f_s that can be acted on the upper block of mass m is, $(f_s)_{max} = \mu N = \mu mg$. If the maximum amplitude is A_{max}, then,

$$(f_s)_{max} = m|a|$$
$$\mu mg = m\omega^2 A_{max}$$
$$\mu mg = m\left(\frac{k}{M+m}\right)A_{max}$$
$$A_{max} = \frac{\mu(M+m)g}{k}$$

Ex. A block of mass m moving with a velocity v collides in elastically with an identical block attached to a spring and sticks to it. Find the amplitude of the resulting simple harmonic motion. Consider all the surfaces to be frictionless.

Sol. Given, The mass of each block is m.
The velocity of one block is v and the velocity of the other block is zero.

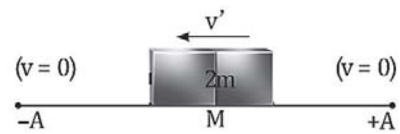


Just before collision Just after collision

If v' is the velocity of the system just after the inelastic collision, then by applying conservation of linear momentum just before and after the collision, we get

$$m(0) + m(v) = 2m(v')$$
$$v' = \frac{v}{2}$$

It is important to note that the collision happened at the mean position of the new SHM because the spring had its natural length just before the collision. If A is the amplitude of the SHM, then the velocity of the system at the mean position will be Aω.



Now, the time period of the SHM performed by the system of mass 2 m is, $T = 2\pi\sqrt{\frac{2m}{k}}$

Hence, the angular frequency will be, $\omega = \sqrt{\frac{k}{2m}}$

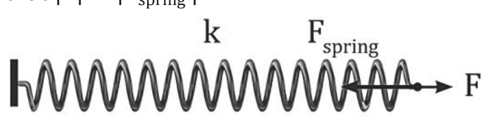
Just after the collision, the system will be at the mean position and the velocity of the system will be $v' = \frac{v}{2}$. Thus,

$$v' = A\omega$$
$$\frac{v}{2} = A\sqrt{\frac{k}{2m}}$$
$$A = \frac{v}{2}\sqrt{\frac{2m}{k}}$$
$$A = v\sqrt{\frac{m}{2k}}$$

Therefore, the amplitude of the resulting simple harmonic motion is $v\sqrt{\frac{m}{2k}}$.

Spring-Mass System

Contemplate a weightless ideal spring with a spring constant k being elongated by an external force F, leading to the emergence of the restoring force. $F_{spring} = -kx$ A force will arise within the spring, with x representing the spring extension. Since the spring is without mass, the net force acting on it is zero. This implies that $|F| = |F_{spring}|$



If l_0 represents the natural length of the spring and Δl is the extension caused by the applied force F, the stress and strain developed in the spring are defined as follows:

Stress $(\sigma) = \frac{F}{A} = \frac{F_{spring}}{A}$

Strain $(\varepsilon) = \frac{\Delta l}{l_0}$

Thus, Young's modulus is defined as follows:

$$Y = \frac{\text{Stress }(\sigma)}{\text{Strain}(\varepsilon)}$$
$$Y = \frac{(\frac{F_{spring}}{A})}{(\frac{\Delta l}{l_0})}$$
$$Y = (\frac{F_{spring}}{A}) \times (\frac{l_0}{\Delta l})$$
$$\frac{F_{spring}}{\Delta l} = \frac{YA}{l_0}$$

Now, since the extension in the spring is, the force is defined as $F_{spring} = k\Delta l$ (taking magnitude only).

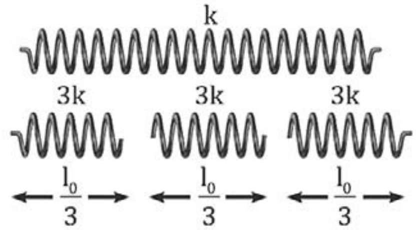
$$\frac{k\Delta l}{\Delta l} = \frac{YA}{l_0}$$

$$k = \frac{YA}{l_0}$$

Since both Y and A are constants for small deformation,

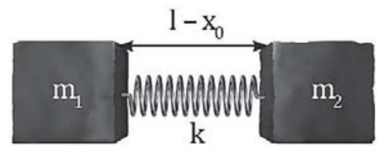
$$k \propto \frac{1}{l_0}$$

Hence, the spring constant (k) exhibits an inverse proportionality to the natural length (l_0) of the spring.
When a spring is divided into 'n' identical segments, the spring constant of each segment will be nk.
Example: If a spring with an initial length of l_0 and spring constant k is divided into three equal parts, where each length becomes $\frac{l_0}{3}$) Subsequently, the spring constant of each segment becomes 3k.



Note: As k denotes the stiffness of the spring, the proportional connection between k and l_0 implies that as the length of the spring increases, the value of k decreases, indicating lower stiffness, and vice versa.

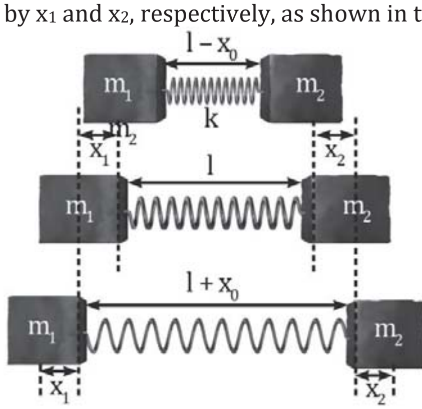
Ex. Two blocks of masses m_1 and m_2 are connected with a spring of natural length l and spring constant k. The system is lying on a smooth horizontal surface. Initially, the spring is compressed by x_0. Show that the blocks will perform SHM about their equilibrium position.



(a) Find the time period of SHM. (b) Find the amplitude of each block.

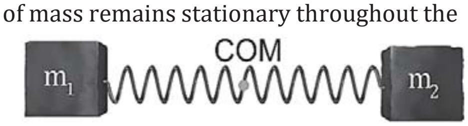
Sol. (a) Consider that the given spring has a natural length l, initial compression x_0, and $(l + x_0)$ is the length after relaxing the compression.

Since the masses are different, their displacements will also be different. Therefore, let us consider m_1 and m_2 are di splaced by x_1 and x_2, respectively, as shown in the figure.

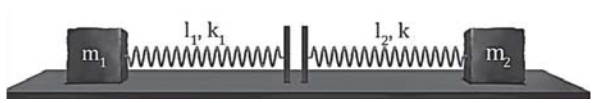


Initially, both masses m_1 and m_2 are at rest, so the centre of mass of the system will also be at rest. Both the masses are just compressed and left, and there is.

With no external force acting on the system, the center of mass remains stationary throughout the entire process. Upon releasing the compressed spring and reaching its natural length (l), the net force acting on the masses is m_1 and m_2 will be zero. Hence, this is an equilibrium position about which m_1 and m_2 execute SHM.



As the center of mass is stationary, the entire spring-mass system can be partitioned into two segments. The center of mass serves as a fixed point, and the two masses undergo individual Simple Harmonic Motion (SHM) with distinct time periods and amplitudes, as depicted in the figure.



Imagine the spring connected to masses m_1 and m_2, with lengths l_1 and l_2, and corresponding spring constants k_1 and k_2, respectively. Therefore,

$$l_1 + l_2 = l \qquad \text{... (1)}$$

Now, by selecting the center of mass of the system as the origin (i.e., adopting the center of mass reference frame), the coordinate of the center of mass becomes zero. This indicates that.

$$m_1 l_1 = m_2 l_2 \qquad \text{... (2)}$$

Since $k \propto \frac{1}{l} \Rightarrow kl =$ Constant, for this case, $kl = k_1 l_1 = k_2 l_2$ ... (3)

The time periods of the blocks of masses m_1 and m_2 are $T_1 = 2\pi\sqrt{\frac{m_1}{k_1}}$ and $T_2 = 2\pi\sqrt{\frac{m_2}{k_2}}$, Respectively.

It is noteworthy that we have determined the time period in the center of mass reference frame, and it is crucial to recall that the time period or time remains independent of the frame of reference, as long as we are within a non-relativistic domain.

By dividing equation (2) by equation (3), we get the following:

$$\frac{m_1 l_1}{k_1 l_1} = \frac{m_2 l_2}{k_2 l_2}$$

$$\frac{m_1}{k_3} = \frac{m_2}{k_3}$$

Therefore, the time periods of both the blocks are the same. Hence,

$$T_1 = T_2 = T$$

Now, by putting the value of l_2 in equation (1) from equation (2), we get the following:

$$l_1 + \frac{m_1 l_1}{m_2} = l \text{[Since from equation (2), we get, } l_2 = \frac{m_1 l_1}{m_2} \text{]}$$

$$l_1\left(1 + \frac{m_1}{m_2}\right) = l$$

$$l_1\left(\frac{m_1 + m_2}{m_2}\right) = l$$

$$l_1 = \left(\frac{m_2}{m_1 + m_2}\right) l$$

By putting this value of l_1 in equation (3), we get

$$kl = k_1 l_1$$

$$kl = k_1\left(\frac{m_2}{m_1 + m_2}\right) l$$

$$k = k_1\left(\frac{m_2}{m_1 + m_2}\right)$$

$$k_1 = \left(\frac{m_1 + m_2}{m_3}\right) k$$

Now, by dividing both sides of the equation by m_1, we get,

$$\frac{k_1}{m_1} = \left(\frac{m_1 + m_2}{m_2 m_1}\right) k$$

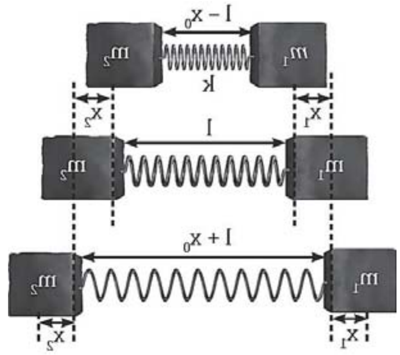
Since $\omega^2 = \frac{k}{m}$ and $\frac{m_1}{k_1} = \frac{m_2}{k_2}$, it can be said that $\omega_1^2 = \omega_2^2 = \omega^2$ (say)

$$\omega^2 = \left(\frac{m_1 + m_2}{m_2 m_1}\right) k$$

$$\omega = \sqrt{\left(\frac{m_1 + m_2}{m_2 m_1}\right) k}$$

Therefore, the time period of the system is given by, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m_1 m_2}{(m_1 + m_2)k}}$

(b) Consider that the blocks of masses m_1 and m_2 are displaced by x_1 and x_2, respectively, as shown in the figure.



Since the initial compression and the final elongation of the spring is x_0, we can write the following as:

$$x_1 + x_2 = x_0 \qquad \text{... (1)}$$

Since we have chosen the centre of mass reference frame, the coordinate of the centre of mass will be zero. It implies that

$$m_1x_1 = m_2x_2 \quad \dots (2)$$

Now, by putting the value of x2 from equation (ii) in equation (i), we get the following

$$x_1 + \frac{m_1x_1}{m_2} = x_0 [\text{ Since from equation (2), we get, } x_2 = \frac{m_1x_1}{m_2}]$$

$$x_1(1 + \frac{m_1}{m_2}) = x_0$$

$$x_1(\frac{m_1+m_2}{m_2}) = x_0$$

$$x_1 = (\frac{m_2}{m_1+m_2})x_0$$

By putting the value of x_1 in equation (1), we get,

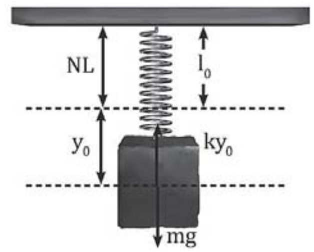
$$x_2 = x_0 - x_1$$

$$x_2 = x_0 - (\frac{m_2}{m_1+m_2})x_0$$

$$x_2 = (\frac{m_1}{m_1+m_2})x_0$$

Therefore, the amplitude of the blocks of masses m_1 and m_2 are $A_1 = \frac{m_2x_0}{m_1+m_2}$ and, $A_2 = \frac{m_1x_0}{m_1+m_2}$, respectively.

Spring-Mass Systems Under Gravity



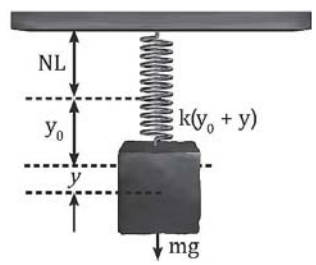
When a spring-mass system is subjected to gravity, meaning if the system is suspended vertically, the mass descends by a short distance due to gravity and reaches the equilibrium state.

This distance is referred to as the equilibrium length in a vertical plane. It represents the natural length of the spring and the slight extension in the spring.

Consider m as the mass, l_0 as the natural length of the spring, and y_0 as the extension of the spring due to gravity at the equilibrium length.

$$ky_0 = mg \quad \dots (1)$$

$$y_0 = \frac{mg}{k} \quad \dots (2)$$



Therefore, if a spring suspends, then it extends by $y_0 = \frac{mg}{k}$ due to gravity.

Now, the spring is extended further down by an amount y as shown in the figure. Therefore, the net extension of the spring is $(y + y_0)$.

The forces acting on the block are as follows:

1. The force of gravity (mg) acting in the downward direction.
2. Restoring force $[k(y + y_0)]$ in the upward direction

Thus, the net downward force acting on the block is as follows:

$$F_{net} = mg - k(y + y_0)$$

$$F_{net} = ky_0 - ky - ky_0 (\text{ Since from equation (1), we get, } mg = ky_0)$$

$$F_{net} = -ky$$

Therefore, the block will undergo Simple Harmonic Motion (SHM) with the following time period:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Alternatively, this can be observed in the following manner:

$$\vec{F}_{net} = \vec{F}_{spring} + \vec{F}_{gravity}$$

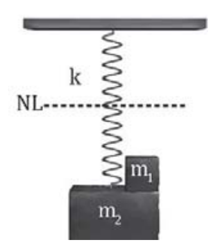
$$F_{net} = -k(y + y_0) + mg \text{ (Taking downward direction to be positive)}$$

$$F_{net} = -ky + (mg - ky_0)$$

Since mg and ky_0 are constants, $(mg - ky_0) = \text{Constant} = c$

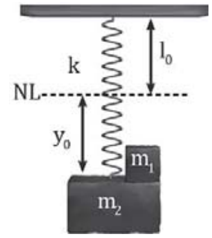
Hence, the net force on the block becomes, $F_{net} = -ky + c$

By applying these forces, we understand that the block undergoes Simple Harmonic Motion (SHM) with the mean position at $\frac{c}{k}$ rather then at zero.



Ex. At the outset, the system depicted in the figure is in a state of equilibrium and rest. Upon removing mass m_1 from m_2, determine the time period and amplitude of the resultant motion.

Sol. Given that the system is initially at rest, let the natural length of the spring be l_0 and initially, the extension in the spring be y_0.



At equilibrium, the forces on the blocks of the combined mass ($m_1 + m_2$) are as follows:

1. Gravitational force [($m_1 + m_2$)g] in the downward direction.
2. Restoring force (k y_0) in the upward direction.

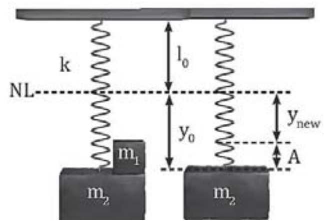
Thus, by balancing the forces, we get,

$$ky_0 = (m_1 + m_2)g$$
$$y_0 = \frac{(m_1+m_2)g}{k}$$

Now, if the block of mass m_1 is removed, then the block of mass m_2 alone executes SHM and the time period is as follows.

$$T = 2\pi\sqrt{\frac{m_2}{k}}$$

As soon as the block of mass m_1 is removed, the spring tries to squeeze its length and the block starts moving in the upward direction from rest at y_0. Let the new extension in the spring be y_{new}.



By balancing the force on this new configuration, we get,

$$ky_{new} = m_2 g$$
$$y_{new} = \frac{m_2 g}{k}$$

Since the block of mass m_2 starts from rest at y_0, this is the extreme position of SHM executed by the block of mass m_2 and ynew is the mean position.

The amplitude is defined as the distance between the mean and extreme positions. Hence, the amplitude of SHM executed by m_2 is as follows.

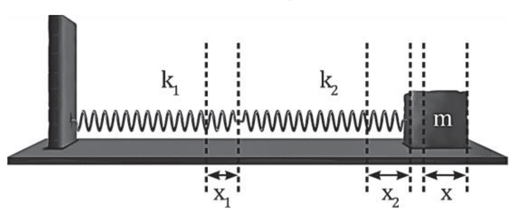
$$A = y_0 - y_{new}$$
$$A = \frac{(m_1+m_2)g}{k} - \frac{m_2 g}{k}$$
$$A = \frac{m_1 g}{k}$$

Therefore, the time period and the amplitude of SHM executed by the block of mass m_2 are $T = 2\pi\sqrt{\frac{m_2}{k}}$ and $A = \frac{m_1 g}{k}$, respectively.

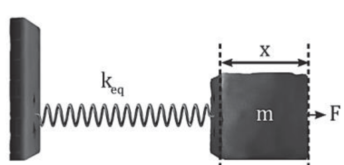
Combination Of Springs

Series Combination

When an external force stretches a series combination of two massless springs with spring constants k_1 and k_2 up to a distance x, the same force is applied to both springs. This results in both springs experiencing the same restoring force. Since a massless spring has a net force of zero, it implies that. $|F_{ext}| = |F_{spring}|$



Take the extensions of the individual springs with spring constants k_1 and k_2 to be x_1 and x_2, respectively. Assume that the entire system is substituted with a single spring of an equivalent spring constant k_{eq}, and the same extension occurs, as illustrated in the figure.



According to the principle of equivalence,

$$x = x_1 + x_2$$

$\frac{F}{k_{eq}} = \frac{F}{k_1} + \frac{F}{k_2}$ (Since the force is same in both the cases)

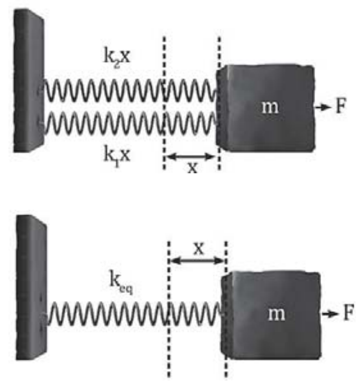
$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

Therefore, when we have a series arrangement of multiple springs, the equivalent spring constant is given by:

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \cdots \ldots \ldots + \frac{1}{k_n}$$

$$\left(\frac{1}{k_{eq}}\right)_{series} = \sum_{i=1}^{n} \frac{1}{k_i}$$

Parallel combination



When an external force F stretches the parallel combination of two massless springs with spring constants k_1 and k_2 up to a distance x, the extension of both springs is identical. In a parallel arrangement of springs, this implies that the extension is uniform across all connected springs, but the force experienced by each spring varies.

Suppose the external force F causes an extension x in the entire system, and the individual springs with spring constants k_1 and k_2 experience forces F_1 and F_2, respectively. Now, consider replacing the entire system with a single spring of an equivalent spring constant keq, resulting in the same extension x due to the force F, as illustrated in the figure.

According to the principle of equivalence,

$$F = F_1 + F_2$$

$k_{eq}\,x = k_1x + k_2x$ (Since the extension is same in both the cases)

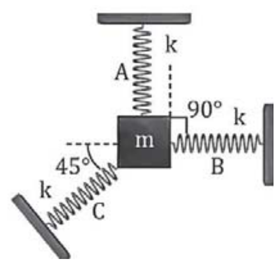
$$k_{mn} = k_1 + k_7$$

Therefore, when multiple springs are arranged in parallel, the equivalent spring constant is determined as follows:

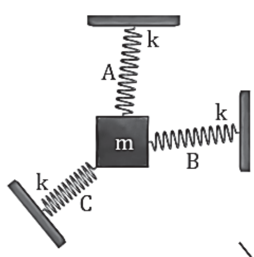
$$k_{eq} = k_1 + k_2 + k_3 + \cdots \ldots .. + k_n$$

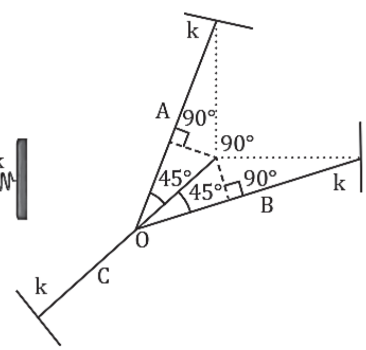
$$(k_{eq})_{parallel} = \sum_{i=1}^{n} k_i$$

Ex. A particle with mass m is connected to three springs, A, B, and C, all having equal force constants k, as depicted in the figure. If the particle is gently pushed against spring C and then released, determine the time period of the resulting oscillation.

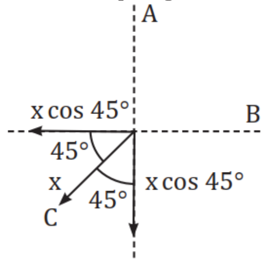


Sol. If spring C is compressed slightly, then springs A and B extend a little but the angles between them can still be assumed to be the same as shown in the figure.

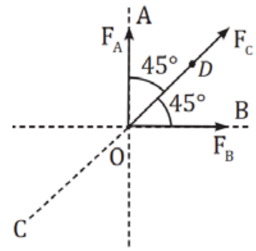




Now, if spring C is compressed by an amount x, then springs A and B are extended by x cos 45° since both the springs make an acute angle of 45° with spring C.



Therefore, the forces acting on springs A and B are F_A and F_B along OA and OB, respectively, and the force acting on spring C is F_C along OD as shown in the figure.



Therefore, the net force on the block is,

$$F_{net} = F_A \cos 45^\circ + F_B \cos 45^\circ + F_C$$

Since both F_A and F_B have the magnitudes equal to kx cos 45° and FC has the magnitude kx,

$$F_{net} = (kx\cos 45^\circ)\cos 45^\circ + (kx\cos 45^\circ)\cos 45^\circ + kx$$

$$F_{net} = 2(kx\cos 45^\circ)\cos 45^\circ + kx$$

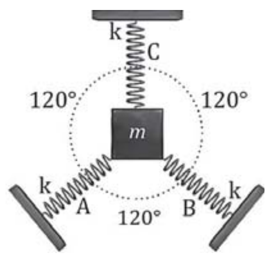
$$F_{net} = 2 \times [(kx \times \frac{1}{\sqrt{2}}) \times \frac{1}{\sqrt{2}}] + kx$$

$$F_{net} = 2 \times (\frac{kx}{2}) + kx$$

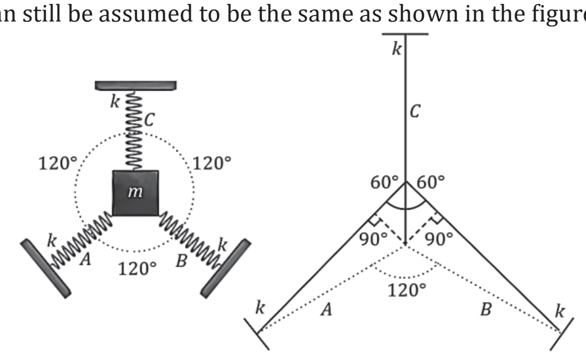
$$F_{net} = 2kx$$

Therefore, the equivalent spring constant of the three springs is $k_{eq} = 2k$. Hence, the time period of the oscillation is as follows: $T = 2\pi\sqrt{\frac{m}{2k}}$

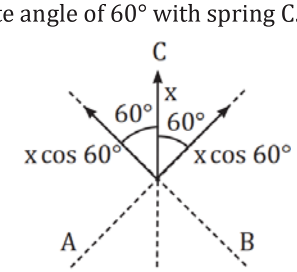
Ex. A particle of mass m is attached to three springs A, B and C of equal force constant k as shown in the figure. Find the time period of the oscillation of the block. Given that the initial angle between the springs is 120°.



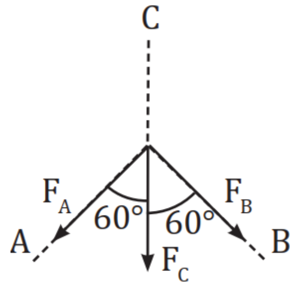
Sol. Consider that spring C is compressed slightly, and springs A and B extend a little, but the angles between them can still be assumed to be the same as shown in the figure



If spring C is compressed by an amount x, then springs A and B are extended by x cos 60° since both the springs make an acute angle of 60° with spring C.



Therefore, the forces acting on springs A and B are F_A and F_B , respectively, that try to pull back the mass to its original position, and the force acting on spring C is F_C that tries to push back the mass to its original position as shown in the figure.



Therefore, the net force on the block is given by,.

$$F_{net} = F_A \cos 60° + F_B \cos 60° + F_C$$

Since both F_A and F_B have the magnitudes equal to kx cos 60° and F_C has the magnitude kx,

$$F_{net} = (kx\cos 60°)\cos 60° + (kx\cos 60°)\cos 60° + kx$$

$$F_{net} = 2(kx\cos 60°)\cos 60° + kx$$

$$F_{net} = 2 \times [(kx \times \frac{1}{2}) \times \frac{1}{2}] + kx$$

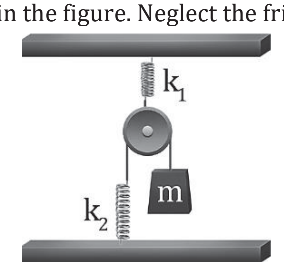
$$F_{net} = 2 \times (\frac{kx}{4}) + kx$$

$$F_{net} = \frac{kx}{2} + kx$$

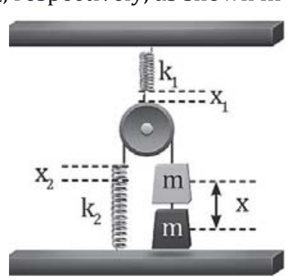
$$F_{net} = \frac{3k}{2}x$$

Therefore, the equivalent spring constant of the three springs is $k_{eq} = \frac{3k}{2}$ hence, the time period of the oscillation is as follows: $T = 2\pi\sqrt{\frac{m}{k_{eq}}} = 2\pi\sqrt{\frac{2m}{3k}}$

Ex. Find the time period of the small oscillations of mass m about its equilibrium position for the given spring mass system, as shown in the figure. Neglect the friction and masses of the springs.



Sol. Since the time period of SHM is independent of its initial position, for this problem also the analysis starts where both the springs are at their natural lengths. Since the mass does not get attached to any spring directly, it cannot be said priori that the mass executes SHM. Thus, initially, we have to prove that the mass executes SHM. Let the mass be displaced by x units down from the position where both the springs are at their natural lengths. As a result, the springs of spring constants k_1 and k_2 get extended by x_1 and k_2, respectively, as shown in the figure.



The forces that act on the mass are as follows:

1. Gravitational force (mg) in the downward direction
2. Tension (T) in the upward direction due to the thread that is attached to the mass, as shown in the FBD of the mass. Thus, the net force acting downwards on the mass is given by,

$$F_{net} = -T + mg \quad \text{... (1)}$$

The forces that act on the pulley are as follows:

1. Tension (T) in the downward direction on the string over the pulley and the string that is connected to the spring of spring constant k_2 and passing over the pulley.

2. Tension (2T) in the upward direction on the spring of spring constant k_1, as shown in the FBD of the pulley. Hence, it can be written as follows:

$$2T = k_1 x_1 \quad \text{... (2)}$$

$$T = k_2 x_2 \quad \text{... (3)}$$

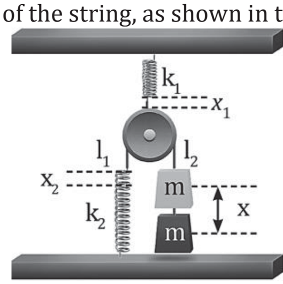
[This tension is due to the spring of spring constant k_2]

The string that is connected between the spring of spring constant k_2 and mass m is inextensible. By using the constraint relation, we get,

$$l_1 + l_2 = \text{Constant}$$

$$\Delta l_1 + \Delta l_2 = 0 \quad \text{... (4)}$$

Where l_1 and l_2 are the lengths of the string, as shown in the figure.



Now, $\Delta l_1 = -(x_1 + x_2)$,Where the negative sign suggests that the length l_1 is decreasing as both the springs extend in the opposite directions, and $\Delta l_2 = (x - x_1)$ is a positive quantity because both the extensions x and x1 are in the same direction and $x > x_1$.

By putting the values of Δl_1 and Δl_2 in equation (4), we get,

$$-(x_1 + x_2) + (x - x_1) = 0$$

$$x = 2x_1 + x_2$$

Now, by putting the values of x_1 and x_2 from equations (2) and (3), we get,

$$x = 2\left(\frac{2T}{k_1}\right) + \left(\frac{T}{k_2}\right)$$

$$x = T\left[\frac{4}{k_1} + \frac{1}{k_2}\right]$$

$$x = T\left[\frac{(4k_2 + k_1)}{k_1 k_2}\right]$$

$$T = \left[\frac{k_1 k_2}{(4k_2 + k_1)}\right]x$$

On substituting the value of T in equation (i), we get,

$$F_{net} = -\left[\frac{k_1 k_2}{(4k_2 + k_1)}\right]x + mg$$

$$F_{net} = -k_{eff}x + mg \quad \text{... (5)}$$

Where k_{eff} = Effective spring constant of the system $= \left[\frac{k_1 k_2}{(4k_2 + k_1)}\right]$

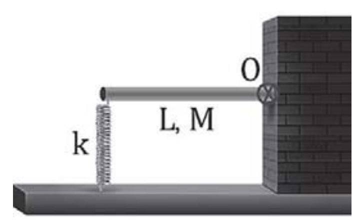
The equation is of the form, hence the mass executes SHM

Thus, the time period of oscillation of mass is given as follows.

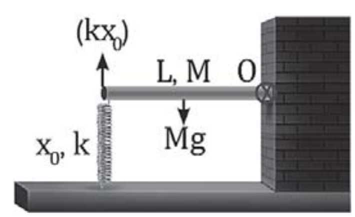
$$T = 2\pi\sqrt{\frac{m}{k_{eff}}} = 2\pi\sqrt{\frac{m(4k_2 + k_1)}{k_1 k_2}}$$

Angular SHM

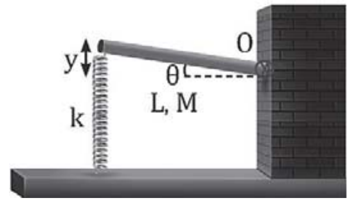
Imagine a lengthy, uniform rod with a length of L and a mass of M, capable of freely rotating in a vertical plane around a horizontal axis (highlighted in red in the figure). The axis is situated at one end, labeled O. Vertically, a spring with a force constant k is connected between one end of the rod and the ground. In the equilibrium state, the rod is parallel to the ground.



The gravitational force (Mg) operates downward through the center of mass (COM) of the rod, producing counterclockwise torque. $[(Mg)\frac{L}{2}]$



Assuming the current length of the spring, already compressed due to the weight of the rod, is x_0, the spring force kx_0 on the rod acts in the upward direction, creating a clockwise torque $[(kx_0)L]$, as depicted in the figure.

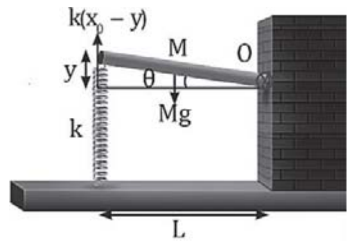


As the rod is in equilibrium, the total force and torque acting on it must be zero. Assuming the clockwise torque as positive and the counterclockwise torque as negative, the overall external torque on the system around point O can be expressed as follows.

$$(\sum\vec{\tau}_{ext}) = 0$$
$$[+(kx_0)L] + [-(Mg)\frac{L}{2}] = 0$$
$$kx_0L - \frac{MgL}{2} = 0$$
$$kx_0 = \frac{Mg}{2}$$

Suppose the spring is stretched vertically in the upward direction by a very small distance y, as shown in the figure, by slightly rotating the rod about point O. Therefore, the angle θ made by the rod at point O is also very small.

Time period of oscillation



As the rod undergoes a minor rotation by a small angle θ, the spring experiences an extension of y. Consequently, the restoring force of the spring becomes $k(x_0 - y)$ in the upward direction, generating a clockwise torque $[k(x_0 - y) L]$ around point O. Simultaneously, the gravitational force (Mg) operates downward through the center of mass (COM) of the rod, producing an anticlockwise torque.$[(Mg)\frac{L}{2}]$ about point O

From the figure, it can also be seen that $\tan\theta = \frac{y}{L}$ ... (2)

We know that the moment of inertia of the rod about point O is given by, $I = \frac{ML^2}{3}$ ... (3)

If α represents the angular acceleration of the rod, then the total torque acting on the rod around point O can be expressed as follows:

$$(\sum\vec{\tau}_{net}) = I\vec{\alpha}$$
$$[+k(x_0 - y)L] + [-\frac{MgL}{2}] = I\alpha$$
$$(kx_0 - \frac{Mg}{2})L - kyL = I\alpha$$

From equation (1), we know that, $kx_0 = \frac{Mg}{2}$

From equation (3), we know that, $I = \frac{ML^2}{3}$

By putting these values in the above equation, we get,

$$0 - kyL = \frac{ML^2}{3}\alpha$$
$$-ky = \frac{ML}{3}\alpha$$
$$\alpha = -(\frac{3k}{M})(\frac{y}{L})$$

$\alpha = -(\frac{3k}{M})\tan\theta[\because$ From equation (2), $\frac{y}{L} = \tan\theta]$

$\alpha = -(\frac{3k}{M})\theta$ ……… (iv)$[\because \theta$ is very small, $\tan\theta \approx \theta]$

The provided equation suggests that the motion of the tip of the rod connected to the spring follows a straight line. Additionally, it is evident that the angular acceleration (α) is directly proportional to the negative of angular displacement (θ), with the negative sign indicating the restoring nature. For linear Simple Harmonic Motion (SHM), the linear acceleration (α) is expressed as $\alpha = -\omega^2x$, where ω represents the angular frequency and x is the linear displacement. Upon comparing this equation with equation (4), it can be concluded that the system exhibits angular SHM with an angular frequency.

$$\omega = \sqrt{\frac{3k}{M}}$$

Thus, the time period of oscillation is, $T = 2\pi\sqrt{\frac{M}{3k}}$

Maximum speed of the displaced end of the rod

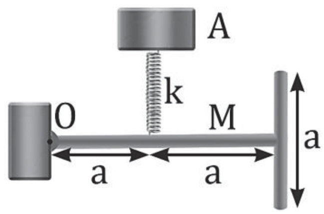
Consider the amplitude (A) of the angular Simple Harmonic Motion (SHM) as θ_0. Therefore, the maximum angular speed can be expressed as follows:

$$(v_{ang})_{max} = \dot{\theta} = A\omega$$
$$\dot{\theta} = \theta_0\sqrt{\frac{3k}{M}}$$

The maximum linear speed is as follows:

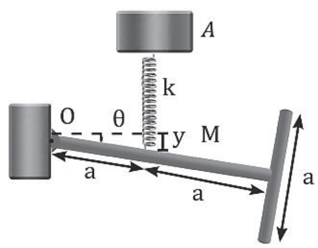
$$v_{max} = \dot{\theta}L$$
$$v_{max} = \theta_0 L\sqrt{\frac{3k}{M}}$$

Ex. A T-bar of uniform cross section and mass M is supported in the vertical plane by hinge O and a spring of force constant k at A. Find the time period of the small oscillations of the bar about point O.



Sol. Suppose the T-bar is slightly displaced by an angle θ. As a consequence, the spring is extended by an amount y in the downward direction, as shown in the figure. From the figure, it can be written as follows:

$$\sin\theta = \frac{y}{a}\left[\because \sin\theta = \frac{\text{Opposite side}}{\text{Hypotenuse}}\right]$$



Since θ is very small, $\sin\theta \approx \theta$

Thus, the relation becomes as following:

$$\theta \approx \frac{y}{a}$$
$$y = a\theta \quad \text{... (1)}$$

Since the total mass of the T-bar of the total length 3a is M and the mass is uniformly distributed over the bar, each part of length a has mass $\frac{m}{3}$.

The net restoring torque of the system about point O is as follows:

$$(\sum\tau_{net})_{restoring} = I\alpha$$
$$(ky)a = I\alpha$$
$$(k \times a\theta)a = I\alpha \quad \text{... (2)}$$
$$[\text{ From equation } (1), y = a\theta]$$

Now, the moment of inertia (I) of the T-bar is composed of two parts.

1. Moment of inertia of the horizontal portion of the bar about point O
2. Moment of inertia of the vertical portion of the bar about point O The moment of inertia of the horizontal portion of the bar about point O is as follows:

$I_1 = \frac{1}{3} \times$ (Mass of the horizontal part) $\times$ (Length of the horizontal part $)^2$

$$I_1 = \frac{1}{3} \times \left(\frac{2M}{3}\right) \times (2a)^2$$
$$I_1 = \frac{8Ma^2}{9}$$

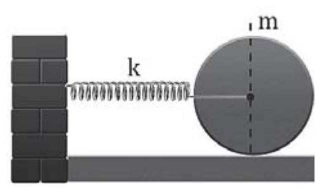
The moment of inertia of the vertical portion of the bar about point O can be found by using the parallel axis theorem as follows

$I_2 = I_{CM} + (\text{ Mass of the vertical part }) \times (\text{ Distance of COM of vertical part from point O})^2$
Now, I_{CM} for the vertical part is,
$I_{CM} = [\frac{1}{12} \times (\text{ Mass of the vertical part }) \times (\text{ Length of the vertical part })^2]$

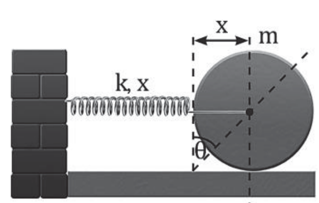
$$I_{CM} = \frac{1}{12} \times (\frac{M}{3}) \times a^2$$
$$I_{CM} = \frac{Ma^2}{36}$$
$$I_2 = \frac{Ma^2}{36} + (\frac{M}{3})(2a)^2$$
$$I_2 = \frac{Ma^2}{36} + \frac{4Ma^2}{3}$$
$$I_2 = \frac{49Ma^2}{36}$$

Hence, the total moment of inertia of the T-bar about point O is as follows:
$$I = I_1 + I_2$$
By substituting this in equation (2), we get the following:
$$(k \times a\theta)a = I\alpha$$
$$ka^2\theta = (I_1 + I_2)\alpha$$
$$ka^2\theta = [\frac{8Ma^2}{9} + \frac{49Ma^2}{36}]\alpha$$
$$k\theta = [\frac{81M}{36}]\alpha$$
$$k\theta = [\frac{9M}{4}]\alpha$$
$$\alpha = [\frac{4k}{9M}]\theta$$
Therefore, the angular frequency of the oscillation of the bar is,
$$\omega = \sqrt{\frac{4k}{9M}} = \frac{2}{3}\sqrt{\frac{k}{M}}$$
Hence, the time period of the oscillation of the bar is,
$$T = \frac{2\pi}{\omega} = 2\pi \times \frac{3}{2}\sqrt{\frac{M}{k}}$$
$$T = 3\pi\sqrt{\frac{M}{k}}$$

Ex. A solid cylinder of mass m is attached to a horizontal spring with spring constant k. The cylinder rolls without slipping along the horizontal plane. Show that the centre of mass of the cylinder executes simple harmonic motion with a period $T = 2\pi\sqrt{\frac{3m}{2k}}$ when it is displaced from the mean position.

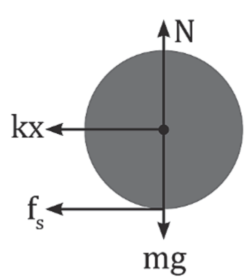


Sol. Suppose the cylinder is displaced in such a way that the centre is displaced by x. Hence, the spring also extends by x, as shown in the figure. The forces that act on the cylinder are as follows:



1. Gravitational force (mg) in the downward direction
2. Normal reaction force N in the upward direction
3. Restoring force due to the spring kx towards the left
4. Static frictional force f s towards the left

If the linear acceleration of the cylinder is 'a' towards the right and the angular acceleration is 'α' in the clockwise direction, then the net force that acts on the cylinder is as follows:
$$-f - kx = ma \quad \text{... (1)}$$



Assuming the radius of the cylinder is r, the net torque on the cylinder about COM of the cylinder is, $\tau_{COM} = f_s r$, and the moment of inertia of the cylinder about the COM is, $I_{COM} = \frac{mr^2}{2}$

Therefore, from the relation $\sum \vec{\tau}_{CoM} = I_{CoM}\vec{\alpha}$, we get,

$$f_s r = \left(\frac{mr^2}{2}\right)\alpha$$

$$f_s = \left(\frac{mr}{2}\right)\alpha$$

Since the cylinder rolls without slipping, we have a constraint relation $a = \alpha r$. On substituting this in equation (2) we get the following.

$$f_s = \frac{m}{2}\left(\frac{a}{\alpha}\right)\alpha$$

$$f_s = \frac{ma}{2}$$

On substituting equation (iii) in equation (1), we get,

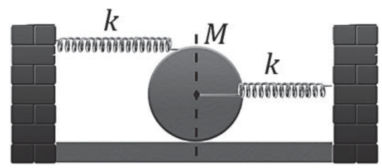
$$-\frac{ma}{2} - kx = ma$$

$$kx = -\frac{3m}{2}a$$

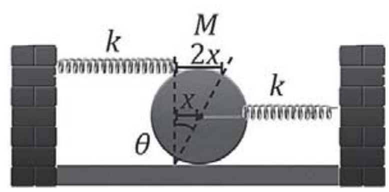
$$a = -\left(\frac{2k}{3m}\right)x$$

Therefore the angular frequency of the oscillation is $\omega = \sqrt{\frac{2k}{3m}}$ hence the time period of the oscillation is, $T = 2\pi\sqrt{\frac{3m}{2k}}$

Ex. A solid uniform cylinder of mass M performs small oscillations in the horizontal plane if slightly displaced from its mean position. The cylinder rolls without slipping along the horizontal plane. Find the time period of the oscillations.

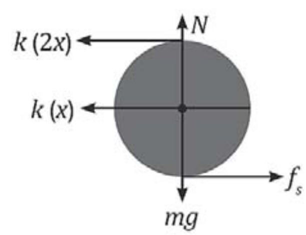


Sol. The cylinder performs small oscillations, which suggests that the springs remain horizontal during the oscillation. Suppose the centre of the cylinder is displaced by x. Since it performs pure rolling, the top point of the cylinder gets displaced by 2x. Thus, the left spring stretches by 2x and the right spring squeezes by x.



The forces that acts on the cylinder are as follows:

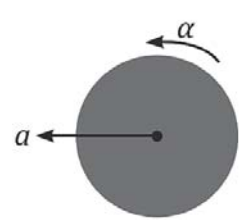
1. Gravitational force (mg) in the downward direction
2. Normal reaction force N in the upward direction
3. Restoring force due to the spring kx towards the left
4. Restoring force due to the spring 2kx towards the left
5. Static frictional force f s towards the right



If the linear acceleration of the cylinder is 'a' towards the left and the angular acceleration is 'α' in the anti-clockwise direction, then the net force acting on the cylinder is as follows:

$$kx + k(2x) - f_s = ma \quad \text{... (1)}$$

Assuming that the radius of the cylinder is r, the net torque on the cylinder about the COM of the cylinder is, $\tau_{COM} = f_s r$, and the moment of inertia of the cylinder about the COM is given as follows: $I_{COM} = \frac{mr^2}{2}$



Therefore, from the relation $\sum \vec{\tau}_{Com} = I_{Com}\vec{\alpha}$, we get,

$$f_s r + k(2x)r = \left(\frac{mr^2}{2}\right)\alpha$$

$$f_s + k(2x) = \left(\frac{mr}{2}\right)\alpha. \quad \text{... (2)}$$

Since the cylinder rolls without slipping, we have constraint relation, $a = \alpha r$.

On substituting this in equation (2), we get the following:

$$f_s + k(2x) = \frac{m}{2}\left(\frac{a}{\alpha}\right)\alpha$$

$$f_s + k(2x) = \frac{ma}{2} \quad \dots (3)$$

By adding equation (3) and equation (1), we get,

$$5kx = ma + \frac{ma}{2}$$

$$kx = \frac{3m}{10}a$$

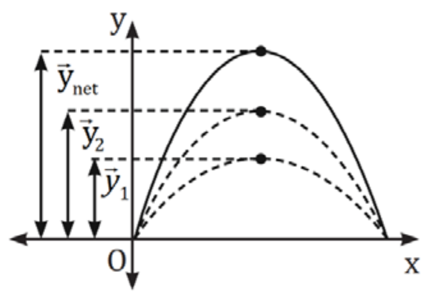
$$a = \left(\frac{10k}{3m}\right)x$$

Therefore, the angular frequency of the oscillation is, $\omega = \sqrt{\frac{10k}{3m}}$

Hence, the time period of the oscillation is, $T = 2\pi\sqrt{\frac{3m}{10k}}$

The Superposition Principle

The combined effect of two or more stimuli is the total of the individual responses that each stimulus would have elicited on its own.



For vector response:

In accordance with the principle of superposition, the overall response ($\vec{r}_{net}$) as a result of all the other individual reactions.($\vec{r}_1, \vec{r}_2, \vec{r}_3 \dots\dots, \vec{r}_n$) will be, $\vec{r}_{net} = \vec{r}_1 + \vec{r}_2 + \vec{r}_3 + \cdots . + \vec{r}_n$

Consider two displacements. $\vec{y}_1$ and $\vec{y}_2$ Occurred due to two distinct factors. The resulting displacement would be, $\vec{y}_{net} = \vec{y}_1 + \vec{y}_2$, as shown in the figure.

For scalar response:

Following the principle of superposition, the overall response (r_{net}) resulting from all other individual reactions ($r_1, r_2, r_3, \dots\dots, r_n$) will be, $r_{net} = r_1 + r_2 + r_3 + \cdots . + r_n$.

Superposition of two SHMs

Same direction (same straight line) and same frequency

Take two simple harmonic motions (SHMs) with identical frequency and direction, generated by two harmonic oscillators, expressed as $x_1 = A_1 \sin(\omega t)$ and $x_2 = A_2 \sin(\omega t + \phi)$, where ϕ represents the phase difference between the two SHMs. When these two SHMs are combined, the resulting response is.

$$x = x_1 + x_2$$

$$x = A_1\sin(\omega t) + A_2\sin(\omega t + \phi)$$

$$x = A_1\sin(\omega t) + A_2[\sin(\omega t)\cos\phi + \cos(\omega t)\sin\phi]$$

$$x = [A_1 + A_2\cos\phi]\sin(\omega t) + [A_2\sin\phi]\cos(\omega t)$$

By denoting $[A_1 + A_2\cos\phi] = A\cos\delta$ and $A_2\sin\phi = A\sin\delta$, we get,

$$x = A\cos\delta\sin(\omega t) + A\sin\delta\cos(\omega t)$$

$$x = A[\sin(\omega t)\cos\delta + \cos(\omega t)\sin\delta]$$

$$x = A\sin(\omega t + \delta)$$

Therefore, superposition of two SHMs of same frequency and direction gives another SHM of same frequency. Now, we assumed,

$$[A_1 + A_2\cos\phi] = A\cos\delta \quad \dots (1)$$

$$A_2\sin\phi = A\sin\delta \quad \dots (2)$$

By squaring and adding these two equations, we get,

$$A^2 = \sqrt{[A_1 + A_2\cos\phi]^2 + [A_2\sin\phi]^2}$$

$$A = \sqrt{[A_1 + A_2\cos\phi]^2 + [A_2\sin\phi]^2}$$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$$

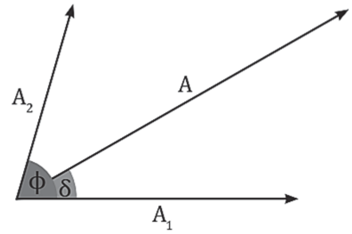
Dividing equation (2) by equation (1), we get,

$$\tan\delta = \frac{A_2\sin\phi}{A_1 + A_2\cos\phi}$$

$$\delta = \tan^{-1}\left(\frac{A_2\sin\phi}{A_1 + A_2\cos\phi}\right)$$

Geometrical Method (Phasor Diagram)

Same straight line and same frequency



Examine two simple harmonic motions (SHMs) represented by $x_1 = A_1 \sin(\omega t)$ and $x_2 = A_2 \sin(\omega t + \phi)$, where ϕ denotes the phase difference between the two SHMs. Upon superposing these two SHMs, the resulting combined response can be expressed as $x = A \sin(\omega t + \delta)$. To determine the unknowns A and δ, consider two phasors, A_1 and A_2, with an angle ϕ between them. These phasors can be added as vectors, as depicted in the figure.

Now, by applying the formula to find the resultant vector, we get,

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$$

$$\delta = \tan^{-1}\left(\frac{A_2\sin\phi}{A_1 + A_2\cos\phi}\right)$$

Same straight line and different frequency

Examine two simple harmonic motions (SHMs) given by $x_1 = A_1 \sin(\omega_1 t)$ and $x_2 = A_2 \sin(\omega_2 t + \phi)$, where ϕ represents the phase difference between the two SHMs. When these two SHMs are combined, the resulting overall response will be,

$$x = x_1 + x_2$$

$$x = A_1\sin(\omega_1 t) + A_2\sin(\omega_2 t)\cos\phi + A_2\cos(\omega_2 t)\sin\phi$$

Therefore, it cannot be written as, $x = A \sin(\omega t + \delta)$ and hence, the resultant motion is not the SHM.

Note: A phasor is a line with a magnitude equal to the amplitude of the simple harmonic motion (SHM), and the angle between two phasors represents the phase difference between two SHMs.

In two perpendicular directions and of same frequency

Consider two SHMs produced by two harmonic oscillators as,

$$x = A_1\sin(\omega t) \quad \text{... (1)}$$

$$y = A_2\sin(\omega t + \phi) \quad \text{... (2)}$$

where ϕ is the phase difference between two SHMs.

By expanding equation (2), we get,

$$y = A_2\sin(\omega t)\cos\phi + A_2\cos(\omega t)\sin\phi$$

Now, by using equation (1), we will eliminate time dependence as follows:

$$y = A_2\left(\frac{x}{A_1}\right)\cos\phi + A_2\sqrt{1 - \sin^2(\omega t)}\sin\phi$$

$$y = A_2\left(\frac{x}{A_1}\right)\cos\phi + A_2\sqrt{1 - \left(\frac{x}{A_1}\right)^2}\sin\phi$$

$$y - \left(\frac{A_2\cos\phi}{A_1}\right)x = A_2\sin\phi\sqrt{1 - \left(\frac{x}{A_1}\right)^2}$$

By squaring both sides of the equation, we get,

$$y^2 - \frac{2A_2\cos\phi}{A_1}xy + \frac{A_2^2\cos^2\phi}{A_1^2}x^2 = A_2^2\sin^2\phi\left(1 - \frac{x^2}{A_1^2}\right)$$

$$y^2 - \frac{2A_2\cos\phi}{A_1}xy + \frac{A_2^2\cos^2\phi}{A_1^2}x^2 = A_2^2\sin^2\phi - \frac{A_2^2\sin^2\phi}{A_1^2}x^2$$

$$y^2 - \frac{2A_2\cos\phi}{A_1}xy + \frac{A_2^2x^2}{A_1^2}(\cos^2\phi + \sin^2\phi) = A_2^2\sin^2\phi$$

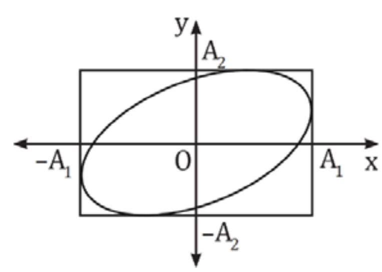
$$y^2 - \frac{2A_2\cos\phi}{A_1}xy + \frac{A_2^2x^2}{A_1^2} = A_2^2\sin^2\phi$$

Dividing both sides of the equation by A_2^2, we get,

$$\frac{y^2}{A_2^2} + \frac{x^2}{A_1^2} - \frac{2xy}{A_1A_2}\cos\phi = \sin^2\phi \quad \text{... (3)}$$

This is the equation of an ellipse as shown in figure.

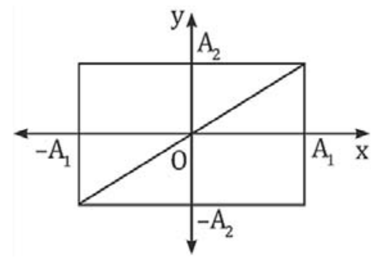
When two SHMs of same frequency but perpendicular to one another are superposed, this type of different figures are obtained depending on the value of A_1, A_2, and ϕ. These figures are known as Lissajous figures.



Case 1 ($\phi = 0°$): By putting $\phi = 0°$ in equation (3), we get the following:

$$\frac{y^2}{A_2^2} + \frac{x^2}{A_1^2} - \frac{2xy}{A_1A_2} \times 1 = 0$$
$$(\frac{x}{A_1} - \frac{y}{A_2})^2 = 0$$
$$y = \frac{A_2}{A_1}x$$

Therefore, the resultant motion will be SHM along a straight line passing through the origin and having a slope $\tan^{-1}(\frac{A_2}{A_1})$ as show in the figure.



Now, by putting $\phi = 0°$ in equation (2), we get,

$$x = A_1\sin(\omega t) \text{ and } y = A_2\sin(\omega t)$$

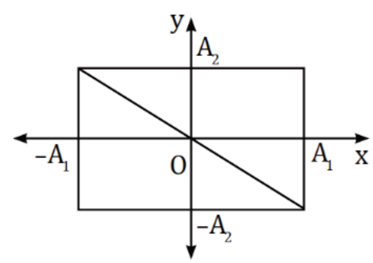
Therefore, the displacement of the particle on this straight line at any time t is,

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(A_1\sin\omega t)^2 + (A_2\sin\omega t)^2}$$
$$r = \sqrt{(A_1^2 + A_2^2)}\sin\omega t$$

Case 2($\phi = \pi$) : By putting $\phi = \pi$ in equation (3), we get the following:

$$\frac{y^2}{A_2^2} + \frac{x^2}{A_1^2} - \frac{2xy}{A_1A_2} \times (-1) = 0$$
$$(\frac{x}{A_1} + \frac{y}{A_2})^2 = 0$$
$$y = -\frac{A_2}{A_1}x$$

Hence, the resulting motion will be a simple harmonic motion (SHM) along a straight line that passes through the origin and has a certain slope. $\tan^{-1}(-\frac{A_2}{A_1})$ as show in the figure.



Now by putting $\phi = \pi$ in equation (2) we get,

$$x = A_1\sin(\omega t) \text{ and } y = A_2\sin(\omega t + \pi)$$

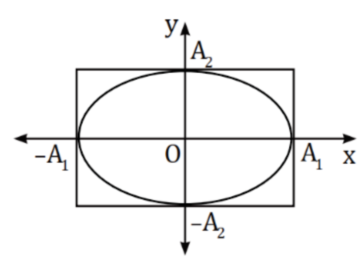
Therefore, the displacement of the particle on this straight line at any time t is.

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{[A_1\sin\omega t]^2 + [A_2\sin(\omega t + \pi)]^2}$$
$$r = \sqrt{[A_1\sin\omega t]^2 + [A_2\sin\omega t]^2}$$
$$r = \sqrt{(A_1^2 + A_2^2)}\sin\omega t$$

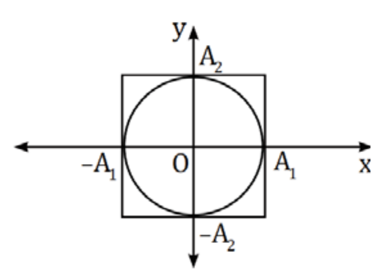
Case 3($\phi = \frac{\pi}{2}$) : By putting $\phi = \frac{\pi}{2}$ in equation (3), we get the following

$$\frac{y^2}{A_2^2} + \frac{x^2}{A_1^2} - \frac{2xy}{A_1A_2} \times (0) = 1$$
$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1$$

Thus, the resulting motion will manifest as an ellipse, as illustrated in the figure. Consequently, the overall motion will deviate from a simple harmonic motion (SHM).



If the semi major axis A_1 and semi minor axis A_2 becomes equal to common value A, i.e., $A_1 = A_2 = A$, then the equation of the ellipse becomes an equation of a circle as given by the equation: $x_2 + y_2 = A^2$. Hence, the resulting motion of two simple harmonic motions (SHMs) with equal amplitude and frequency in perpendicular directions exhibits a difference. $\frac{\pi}{2}$The phase represents a motion that is uniformly circular.



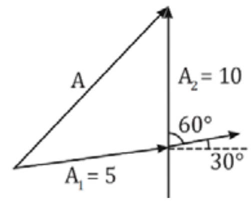
Ex. Particle is subjected to two simple harmonic motions, $x_1 = 5\sin(\omega t + 30°)$ and $x2 = 10\cos(\omega t)$. Find the amplitude of the resultant SHM.

Sol. Given that

$$x_1 = 5\sin(\omega t + 30°) \quad \text{... (1)}$$

$$x_2 = 10\cos(\omega t) = 10\sin(\omega t + 90°) \quad \text{... (2)}$$

As both SHMs share the same frequency and direction, the resultant amplitude can be determined through the aid of a phasor diagram.



Hence, the amplitude of resultant SHM will be,

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$$

and we have, $A_1 = 5, A_2 = 10$ and $\phi = 60°$

$$A = \sqrt{5^2 + 10^2 + 2 \times 5 \times 10 \times \cos(60°)}$$

$$A = \sqrt{25 + 100 + 50}$$

$$A = \sqrt{175}$$

$$A = \sqrt{5^2 \times 7}$$

$$A = 5\sqrt{7}$$

Ex. A particle is subjected to two simple harmonic motions in the same direction having equal amplitude and equal frequency. If the resultant amplitude is equal to the amplitude of the individual motions, find the phase difference between the individual motions.

Sol. Suppose the two SHMs are:

$$x_1 = A\sin(\omega t) \quad \text{... (1)}$$

$$x_2 = A\sin(\omega t + \phi) \quad \text{... (2)}$$

If the amplitude of the resulting simple harmonic motion (SHM) is likewise A, the phase difference between the individual motions can be determined through the following analysis.

The amplitude of resultant motion is,

$$A = \sqrt{A^2 + A^2 + 2A^2\cos\phi}$$

$$A = \sqrt{2A^2(1 + \cos\phi)}$$

Squaring both sides of the equation, we get,

$$A^2 = 2A^2(1 + \cos\phi)$$

$$1 + \cos\phi = \frac{1}{2}$$

$$\cos\phi = -\frac{1}{2}$$

$$\cos\phi = \cos(\frac{2\pi}{3})$$

$$\phi = \frac{2\pi}{3}$$

Therefore, the phase difference between the individual motions is $\frac{2\pi}{3}$.

Ex. Find the amplitude of the simple harmonic motion obtained by combining the motions $x_1 = (2.0\text{cm})\sin(\omega t)$ and $x_2 = (2.0\text{cm})\sin(\omega t + \frac{\pi}{3})$

Sol. Given that,

$$x_1 = (2.0\text{ cm})\sin(\omega t) \quad \text{... (1)}$$

$$x_2 = (2.0\text{cm})\sin(\omega t + \frac{\pi}{3}) \quad \text{... (2)}$$

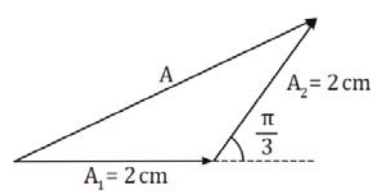
It is evident that both SHMs share the same frequency and direction. Consequently, the resulting amplitude can be determined using a phasor diagram.

Hence, the resultant amplitude will be,.

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$$

And we have, $A_1 = 2\text{cm}, A_2 = 2\text{cm}$ and $\phi = \frac{\pi}{3}$.

$$A = \sqrt{2^2 + 2^2 + 2 \times 2 \times 2 \times \cos(\frac{\pi}{3})}$$



$$A = \sqrt{2^2 + 2^2 + 2 \times 2 \times 2 \times \frac{1}{2}}$$

$$A = \sqrt{2^2 + 2^2 + 2 \times 2}$$

$$A = \sqrt{3 \times 2^2}$$

$$A = 2\sqrt{3}$$

$$A = 3.464\text{ cm}$$