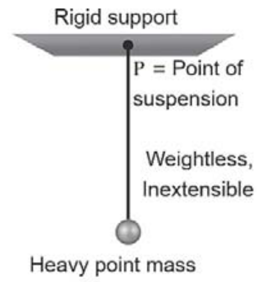
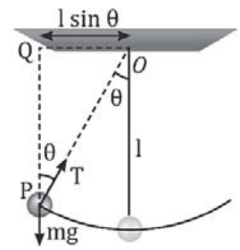
SIMPLE PENDULUM



A simple pendulum comprises a weightless and inextensible ideal string along with a heavy point mass. The point mass is suspended from the string at a rigid support, also referred to as the point of suspension.

The point mass undergoes periodic and oscillatory motions. However, does it follow a simple harmonic motion? Let's investigate.

Assume a pendulum with a length of l is displaced by an angle θ from its equilibrium position, and upon release, the trajectory of the point mass is depicted by a yellow arc in the Free Body Diagram (FBD) of the point mass.



Upon release, the point mass initiate's oscillatory motion along the yellow arc, driven by the restoring force exerted by gravity, which is mg.

Two forces are acting on the point mass, which are:

1. Gravitational force, mg
2. Tension (T) due to the string

Hence, the total torque acting on the point mass around point O can be expressed as: $\sum \tau_{net} = I_0 \alpha$

Where = Moment of inertia of the point mass about point $O = ml^2$

Since tension is passing through point about which the net torque is measured, the torque due to the tension is zero and the torque due to the gravitational force is mg(lsin θ). Therefore,

$$mg(l\sin\theta) = (ml^2)\alpha$$

$$\alpha = \frac{g}{l}\sin\theta$$

Since is not proportional to , the motion of a pendulum is not SHM. However, if we consider small oscillations, then $\sin\theta \approx \theta$

$$\alpha = \frac{g}{l}\theta$$

Hence, under small oscillations, the pendulum execute SHM with angular frequency $\omega = \sqrt{\frac{g}{l}}$ and time period, $T = 2\pi\sqrt{\frac{l}{g}}$

Note: While the motion of a simple pendulum is consistently periodic and oscillatory, it strictly follows Simple Harmonic Motion (SHM) only when its angular displacement, θ, is minimal. Therefore, the formula for the time period of SHM performed by a simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$ is valid only for small values of θ.

Second's pendulum

1. A seconds pendulum refers to a simple pendulum with a time period of two seconds.
2. Each swing, from one end to the other, takes precisely one second in either direction.

Simple pendulum in accelerating frame

Consider two simple pendulums: one is in the ground frame (inertial frame) observed by observer O_1, and the other is within an accelerating lift (non-inertial frame) observed by observer O_2.

For observer O_1, situated in the ground frame, the motion of the simple pendulum in the lift will appear as a zig-zag motion, as depicted in the figure.

Hence, it's crucial to emphasize that the motion of a simple pendulum in an accelerating lift, as observed by an observer in an inertial frame, is neither periodic and oscillatory nor does it adhere to Simple Harmonic Motion (SHM).

For observer O_2 within the lift, the motion of the simple pendulum suspended in the lift will appear as Simple Harmonic Motion (SHM) with the time period.

$$T = 2\pi\sqrt{\frac{l}{g_{eff}}}$$

Where $g_{eff} = |\vec{g} - \vec{a}|$ at the mean position of the pendulum and represents the acceleration of the non-inertial frame in which the pendulum is suspended.

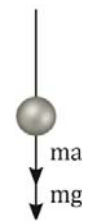
Considering the presence of a pseudo force, $\vec{F}_{pseudo} = -m\vec{a}_{frame}$, with a pseudo force in the non-inertial frame opposing the frame's acceleration, g transforms into g_{eff} in the formula of the time period of the simple pendulum suspended in a non-inertial frame.

Frame accelerating upwards:

In the scenario where the non-inertial frame experiences an upward acceleration a, the pseudo acceleration will be directed downward. Consequently, the effective acceleration will be $g_{eff} = (g + a)$ Vector ally, this can be written as follows:

Consider the non-inertial frame ascending with acceleration, $\vec{a}_{frame} = \vec{a}$

Hence, the pseudo acceleration will be $\vec{a}_{pseudo} = -\vec{a}_{frame} = -\vec{a}$ both the pseudo force and the gravitational force acting on the bob of the pendulum will be directed downward.



Therefore, the effective acceleration will be,

$$g_{eff} = |\vec{g} - \vec{a}| = g - (-a) = g + a$$

The time period of the oscillation will be, $T = 2\pi\sqrt{\frac{l}{g+a}}$

Frame accelerating downwards:

In the event that the non-inertial frame undergoes a downward acceleration a, the pseudo acceleration will be directed upward. Consequently, the effective acceleration will be $g_{eff} = (g - a)$

Hence, the oscillation's time period will be, $T = 2\pi\sqrt{\frac{l}{g-a}}$

Note: Simple Harmonic Motion (SHM) in the motion of a simple pendulum will occur exclusively in frames of reference where the point of suspension is stationary.

Ex. A simple pendulum of length 1 ft suspended from the ceiling of an elevator takes $\frac{\pi}{3}s$ to complete one oscillation. Find the acceleration of the elevator. (Take g = 32 ft s^{-2}).

Sol. We have,

Length of the string of the simple pendulum l = 1ft

Time period of oscillation $T = \frac{\pi}{3}s$

Gravitational acceleration $g = 32fts^{-2}$

Suppose that the lift is moving with acceleration a therefore the time period of the pendulum is.

$$2\pi\sqrt{\frac{l}{g+a}} = \frac{\pi}{3}$$

$$\sqrt{\frac{l}{g+a}} = \frac{1}{6}$$

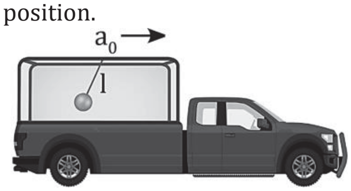
$$\frac{l}{g+a} = (\frac{1}{6})^2$$

$$\frac{1}{32+a} = \frac{1}{36}$$

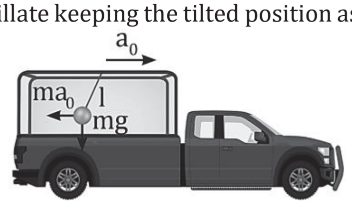
$$a = 36 - 32$$

$$a = 4 \text{ ft s}$$

Ex. A pendulum is suspended from the ceiling of a truck accelerating uniformly on a horizontal road. If the acceleration is a0 and the length of the pendulum is l, find the time period of the small oscillations about the mean position.



Sol. As soon as the truck starts moving with acceleration a0, the pendulum suspended from the ceiling of the truck becomes tilted from its mean position because of the pseudo force as shown in the figure. Now, the pendulum will oscillate keeping the tilted position as the mean position of the oscillation.

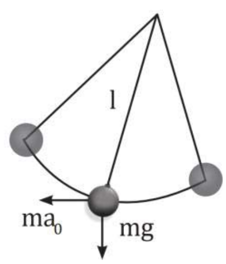


Therefore, the effective acceleration will be,

$$g_{eff} = |\vec{g} - \vec{a}|$$

$$g_{eff} = \sqrt{g^2 + a_0^2 - 2ga_0 \cos 90^\circ} (\because \text{ the angle between } \vec{g} \text{ and } \vec{a}_0 \text{ is } 90^\circ.)$$

$$g_{eff} = \sqrt{g^2 + a_0^2}$$

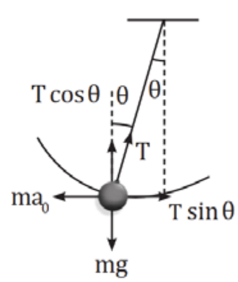


Therefore, the time period of the pendulum will be,

$$T = 2\pi \sqrt{\frac{l}{g_{eff}}}$$

$$T = 2\pi \frac{\sqrt{l}}{(g^2 + a_0^2)^{\frac{1}{4}}}$$

Now, the angle by which the pendulum is tilted due to the acceleration of the truck can also be found. Let the angle be θ as shown in the FBD of the pendulum.



If T is the tension in the string of the pendulum, then on balancing the force on the bob of the pendulum, we get the following:

$$T \sin\theta = ma_0 \quad \text{... (1)}$$

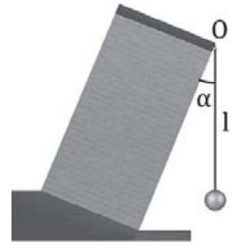
$$T \cos\theta = mg \quad \text{... (2)}$$

Dividing equation (1) by equation (2), we get,

$$\tan\theta = \frac{a_0}{g}$$

$$\theta = \tan^{-1}\left(\frac{a_0}{g}\right)$$

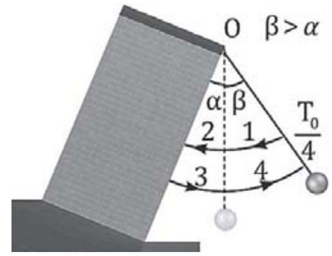
Therefore, the angle by which the pendulum is tilted from its mean position due to the acceleration of the truck is $\theta = \tan^{-1}(\frac{a_0}{g})$.



Ex. A ball is suspended by a thread of length l at point O on a wall, forming a small angle α with the vertical. The thread along with the ball is then deviated through a small angle β ($\beta > \alpha$) and set free. Assuming the collision of the ball against the wall to be perfectly elastic, find the oscillation period of such a pendulum.

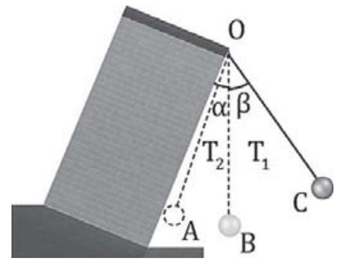
Sol. If the angle of deviation β is smaller than the angle α, then the pendulum will execute SHM without any interruption due to the wall.

Hence, the time period of the SHM will be, $T_0 = 2\pi\sqrt{\frac{l}{g}}$.



However, since the angle of deviation β is greater than the angle α, the pendulum will be interrupted due to the collision of the bob with the wall and since the collision is perfectly elastic, the pendulum will still execute SHM.

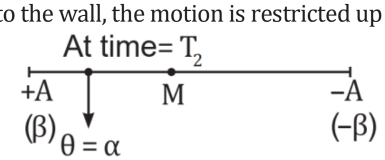
Now, if we divide the interrupted SHM in four parts as shown in the figure and consider the time period of the free swinging (uninterrupted) SHM as T_0, then the time period of parts 1 and 4 will be $\frac{T_0}{4}$ Let the time period of parts 2 and 3 be T.



Let A, B, and C be the interrupted extreme position, mean position, and extreme position of the SHM being executed as shown in the figure. Therefore, the time in going from C to B is, $T_1 = \frac{T_0}{4}$ and the time in going from B to A is, T2 = T. The equation of the angular displacement of the bob from mean position B is,

$$\theta = \beta\sin(\omega t) \quad \dots (1)$$

The motion of the bob can be considered as linear SHM as shown, where M is the mean position. If the motion is uninterrupted, then the bob will execute an uninterrupted SHM with amplitude β but as the motion is interrupted due to the wall, the motion is restricted up to $\theta = \alpha$, as shown in the figure.



Since the bob takes time T_2 to reach $\theta = \alpha$ from the mean position, from equation (1), we get.

$$\alpha = \beta\sin(\omega T_2)$$

$$T_2 = \frac{1}{\omega}\sin^{-1}(\frac{\alpha}{\beta})$$

Since the time period is $T_0 = 2\pi\sqrt{\frac{l}{g}}$ and the angular frequency is $\omega = \sqrt{\frac{g}{l}}$

$$T_2 = \sqrt{\frac{l}{g}}\sin^{-1}(\frac{\alpha}{\beta})$$

Now, the time period of the interrupted SHM is,

$$T = 2T_1 + 2T_2$$

$$T = 2(\frac{T_0}{4}) + 2 \times \sqrt{\frac{l}{g}}\sin^{-1}(\frac{\alpha}{\beta})$$

$$T = \frac{T_0}{2} + 2 \times \sqrt{\frac{l}{g}}\sin^{-1}(\frac{\alpha}{\beta})$$

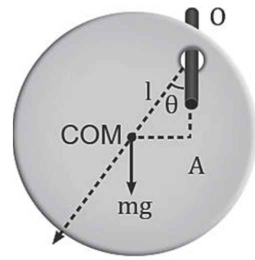
$$T = \frac{1}{2} \times 2\pi\sqrt{\frac{l}{g}} + 2 \times \sqrt{\frac{l}{g}}\sin^{-1}(\frac{\alpha}{\beta})$$

$$T = 2\sqrt{\frac{l}{g}}[\frac{\pi}{2} + \sin^{-1}(\frac{\alpha}{\beta})]$$

Compound Pendulum

When a rigid body is hung from a fixed point and undergoes oscillations around the axis passing through that point of suspension, it is referred to as a compound pendulum or physical pendulum.
In the state of equilibrium, the Center of Mass (COM) of the rigid body will align with the vertical line passing through the point of suspension. This alignment ensures that the torque resulting from the gravitational force around the point of suspension becomes zero.

Time period



Assume the rigid body is shifted by a slight angle θ from the mean position, as illustrated in the figure.
Therefore, the torque about point O is,

$$\tau_0 = (mg) \times l\sin\theta$$
$$I_{Hinge}\,\alpha = mg \times l\sin\theta$$
$$\alpha = \left(\frac{mgl}{I_{Hinge}}\right)\sin\theta$$

Now, for small θ, $\sin\theta \approx \theta$

$$\alpha = \left(\frac{mgl}{I_{Hinge}}\right)\theta$$

Consequently, the rigid body will undergo Simple Harmonic Motion (SHM), and the angular frequency of this SHM will be $\omega = \sqrt{\frac{mgl}{I_{Hinge}}}$

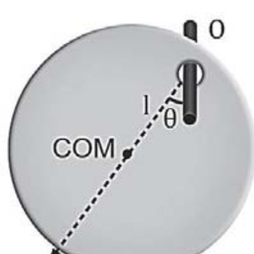
Therefore, the period of the Simple Harmonic Motion (SHM) will be $T = 2\pi\sqrt{\frac{I_{Hinge}}{mgl}}$ where is the distance between the point of suspension and the Center of Mass (COM) of the rigid body, and I_{Hinge} represents the moment of inertia of the rigid body about the point of suspension.
Now, employing the parallel axis theorem, the moment of inertia of the rigid body about the point of suspension can be expressed as , I_{Hinge} , can be written as $I_{Hinge} = I_{COM} + ml^2$, and if k denotes the radius of gyration about the Center of Mass (COM), $I_{CM} = mk^2$ Thus, the moment of inertia of the rigid body about the point of suspension is $I_{Hinge} = mk^2 + ml^2$.
Therefore, the time period becomes,

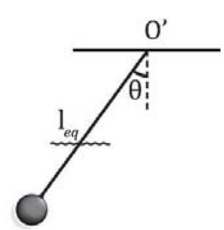
$$T = 2\pi\sqrt{\frac{I_{Hinge}}{mgl}}$$
$$T = 2\pi\sqrt{\frac{(mk^2+ml^2)}{mgl}}$$
$$T = 2\pi\sqrt{\frac{(k^2+l^2)}{gl}}$$
$$T = 2\pi\sqrt{\frac{\left(\frac{k^2+l^2}{l}\right)}{g}}$$

Now, by comparing the time period expression of a compound pendulum with that of a simple pendulum having a length l_{eq} equivalent to the same time period, we can determine the equivalent length of the simple pendulum as.

$$l_{eq} = \frac{k^2+l^2}{l}$$

Hence, the time period of a physical pendulum is identical to that of a simple pendulum with a length l_{eq}.





Variation of time period with length

The radius of gyration, k, and remains constant as the moment of inertia of a rigid body about its Center of Mass (COM) is unchanged. However, it is possible to alter the length of the pendulum. Upon differentiating the time period with respect to the length l and setting it equal to zero, we obtain the following:

$$\frac{dT}{dl} = 0$$

$$\frac{d}{dl}\left(2\pi\sqrt{\frac{(\frac{k^2+l^2}{l})}{g}}\right) = 0$$

$$\frac{d}{dl}\left(\sqrt{(\frac{k^2+l^2}{l})}\right) = 0$$

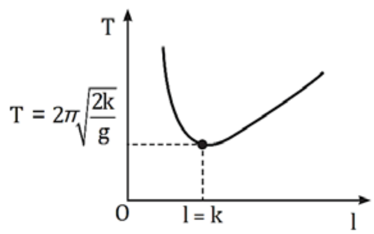
$$\frac{1}{2}\left(\frac{k^2+l^2}{l}\right)^{-\frac{1}{2}} \times \frac{d}{dl}\left(\frac{k^2+l^2}{l}\right) = 0$$

$$\frac{d}{dl}\left(\frac{k^2}{l} + l\right) = 0$$

$$-\frac{k^2}{l^2} + 1 = 0$$

$$l = k$$

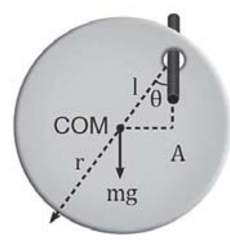
Hence, the time period reaches its minimum when $l = k$, and the minimum value of the time period is: $T_{min} = 2\pi\sqrt{\frac{2k}{g}}$ The figure depicts the variation of the time period.



It is evident from the figure that, except at the minimum time period , T_{min} there are always two positions for suspending the physical pendulum to achieve a specific time period.

Note: The time period will not reach a maximum at $l = k$. If a compound pendulum is suspended through its Center of Mass (COM), meaning $l = k$, the pendulum will be in neutral equilibrium, resulting in an infinite time period. Therefore, there is no upper limit for the time period of a compound pendulum.

Ex. A uniform disc of radius r is to be suspended through a small hole made in the disc. Find the minimum possible time period of the disc for small oscillations. What should be the distance of the hole from the centre for it to have a minimum time period?



Sol. Let the radius of gyration be k.

We know that the moment of inertia of a disc about an axis passing through its COM is,

$$I_{COM} = \frac{mr^2}{2}, \text{ and } I_{COM} = mk^2$$

Therefore, by equating these two expressions, we get,

$$mk^2 = \frac{mr^2}{2}$$

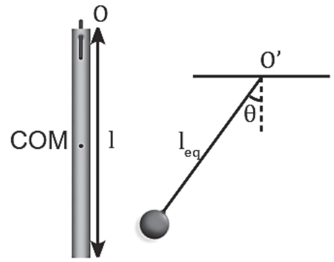
$$k^2 = \frac{r^2}{2}$$

$$k = \frac{r}{\sqrt{2}}$$

The minimum possible time period of the disc for small oscillations is, $T_{min} = 2\pi\sqrt{\frac{2k}{g}} = 2\pi\sqrt{\frac{r\sqrt{2}}{g}}$

and the distance of the hole from the centre for it to have a minimum time period is, $l = k = \frac{r}{\sqrt{2}}$

Ex. A uniform rod of length l is suspended by an end and is made to undergo small oscillations. Find the length of the simple pendulum having a time period equal to that of the rod.



Sol. We know that the time period of a physical pendulum, is $T = 2\pi\sqrt{\frac{I}{mgl_0}}$ where I is the moment of inertia about the point of suspension and l 0 is the distance between the COM and the point of suspension $= l - \frac{l}{2} = \frac{l}{2}$.

Therefore, the moment of inertia of the rod about the point of suspension will be, $I = \frac{ml^2}{3}$.

If the length of the simple pendulum is l_{eq}, then its time period will be, $T = 2\pi\sqrt{\frac{l_{eq}}{g}}$ Therefore, by equating these two time periods, we get,

$$2\pi\sqrt{\frac{I}{mgl_0}} = 2\pi\sqrt{\frac{l_{eq}}{g}}$$

$$\frac{I}{ml_0} = \frac{l_{eq}}{g}$$

$$\frac{\left(\frac{ml^2}{3}\right)}{m\left(\frac{l}{2}\right)} = \frac{l_{eq}}{g}$$

$$l_{eq} = \frac{2l}{3}$$

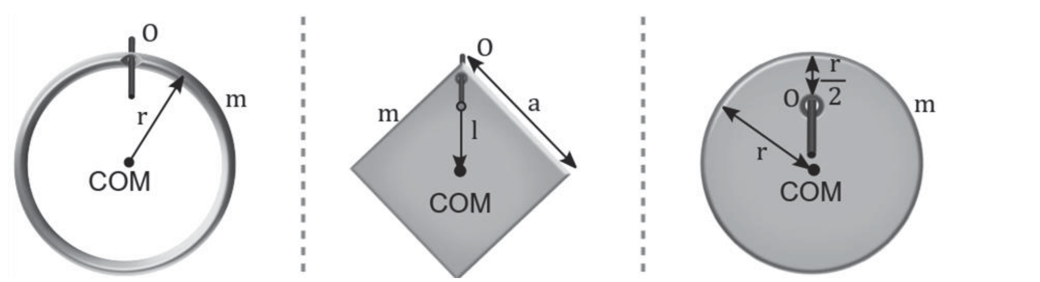
The length of the simple pendulum having the time period equal to that of the rod will be $\frac{2l}{3}$.

Ex. Determine the time period of small oscillations for the following setups:

(a) A ring with mass m and radius r hanging from a point on its outer edge.

(b) A uniformly distributed square plate with side length a hanging from one of its corners

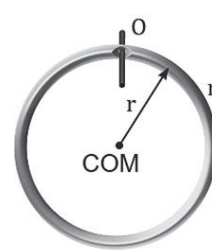
(c)A uniform disc with mass m and radius r hanging from a single point $\frac{r}{2}$ away from the center.



Sol. **(a)** The moment of inertia of the ring about the COM is, I COM = mr^2. Thus, the moment of inertia of the ring about point O is, $I = I_{COM} + mr^2 = mr^2 + mr^2 = 2mr^2$

The distance of the COM from the point of suspension is, $l = r$

Therefore, the time period is,

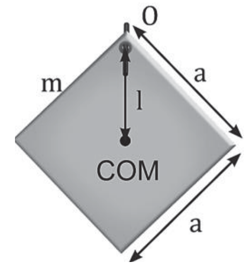


$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

$$T = 2\pi\sqrt{\frac{2mr^2}{mgr}}$$

$$T = 2\pi\sqrt{\frac{2r}{g}}$$

(b) Since the square plate is suspended at one corner, the distance between the COM and the point of suspension is half of the length of the diagonal of the square.



Therefore, $l = \frac{\sqrt{2}a}{2} = \frac{a}{\sqrt{2}}$

Now, the moment of inertia of a uniform square plate about its COM is $I_{COM} = \frac{ma^2}{6}$ therefore, the moment of inertia of the plate about the point of suspension (O) is,

$$I = \frac{ma^2}{6} + m(\frac{a}{\sqrt{2}})^2 = \frac{ma^2}{6} + \frac{ma^2}{2}$$

$$I = \frac{2ma^2}{3}$$

The time period of the oscillation is.

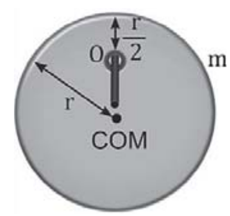
$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

$$T = 2\pi\sqrt{\frac{(\frac{2ma^2}{3})}{mg(\frac{a}{\sqrt{2}})}}$$

$$T = 2\pi\sqrt{\frac{2\sqrt{2}a}{3g}}$$

$$T = 2\pi\sqrt{\frac{\sqrt{8}a}{3g}}$$

(C) The distance of the COM from the point of suspension is, $l = \frac{r}{2}$.



The moment of inertia of the disc about the COM is, $I_{COM} = \frac{mr^2}{2}$ thus, the moment of inertia of the disc about point O is,

$$I = \frac{mr^2}{2} + m(\frac{r}{2})^2$$

$$I = \frac{3mr^2}{4}$$

Therefore the time period of oscillation will be.

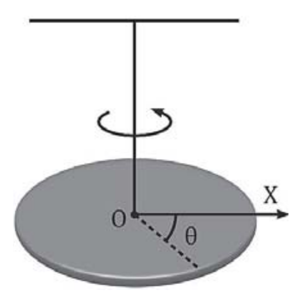
$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

$$T = 2\pi\sqrt{\frac{(\frac{3mr^2}{4})}{mg(\frac{r}{2})}}$$

$$T = 2\pi\sqrt{\frac{3r}{2g}}$$

Torsional Pendulum

Consider an elongated object suspended by a light, massless thread. When the body is rotated around the thread as the axis of rotation, with a small angle, θ, it is noted that the restoring torque produced by the pendulum's thread is directly proportional to the angle of rotation θ, i.e.,



$\tau \propto -\theta$ (The negative sign refers to the restoring torque)

$\tau = -C\theta$ Where is the torsional constant or twisting coefficient.

Now, if the angular acceleration is , then $\tau = I\alpha$

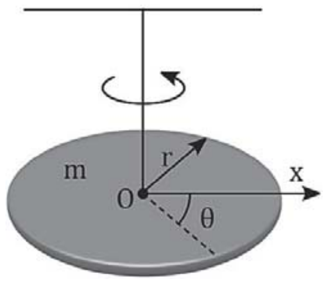
Therefore, $I\alpha = -C\theta$

$$\alpha = -(\frac{C}{I})\theta$$

Therefore, the elongated body undergoes torsional oscillations with angular frequency, $\omega = \sqrt{\frac{C}{I}}$

Hence, the period of the torsional oscillation is, $T = 2\pi\sqrt{\frac{I}{C}}$

Ex. A uniform disc of mass m and radius r is suspended through a wire attached to its centre. If the time period of the torsional oscillations is T, what is the torsional constant of the wire?



Sol. We know that the time period of torsional oscillations is,

$$T = 2\pi\sqrt{\frac{I}{C}}$$

Squaring both the sides of the equation, we get,

$$T^2 = 4\pi^2\left(\frac{I}{C}\right) \quad \dots (1)$$

Now, the moment of inertia of a uniform circular disc about an axis through its COM is $I = \frac{mr^2}{2}$. On substituting this in equation(1), we get,

$$T^2 = \frac{4\pi^2}{C}\left(\frac{mr^2}{2}\right)$$

$$C = \frac{2\pi^2 mr^2}{T^2}$$

Therefore, the torsional coefficient of the wire is, $C = \frac{2\pi^2 mr^2}{T^2}$