Chapter 20

SHM & Oscillations

INTRODUCTION TO SHM

Periodic Motion

When an object consistently retraces its path at regular intervals, its movement is considered periodic. The shortest time span required for the motion to replicate itself is referred to as the time period of the periodic motion.
A body engaged in periodic motion reproduces its movement, signifying that it returns to the identical position with the same velocity and acceleration along a predetermined path after a particular interval of time, known as the time period.

Oscillatory Motion

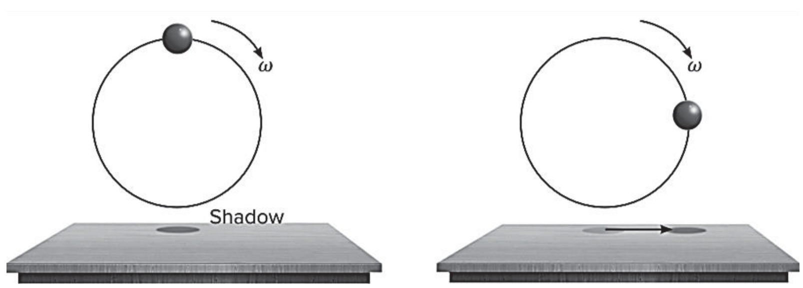
Oscillatory motion is described as the movement of a body back and forth along the same path around a fixed position.
While all oscillatory motions exhibit periodicity, it's important to note that not all periodic motions are oscillatory. For a motion to be considered oscillatory, the body must engage in back-and-forth movement around a fixed point. For instance, uniform circular motion qualifies as periodic but does not fall under the category of oscillatory motion.

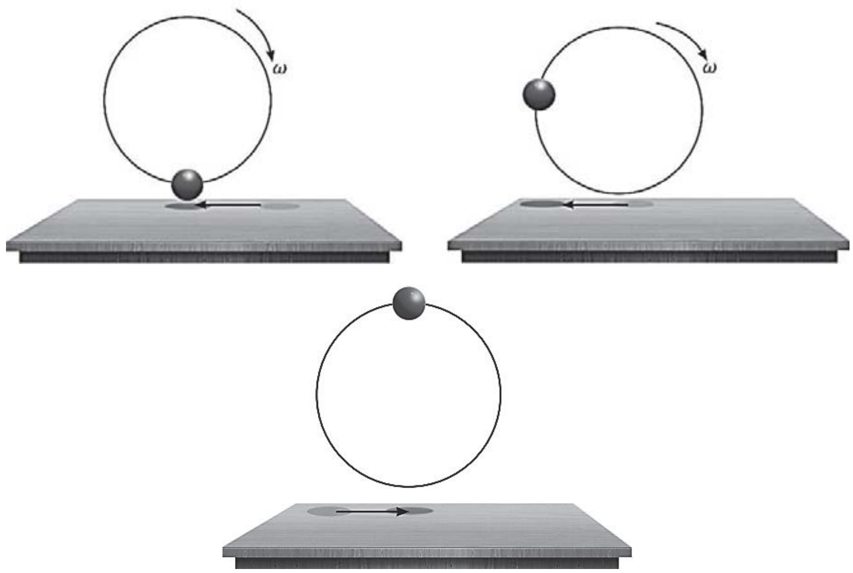
Simple Harmonic Motion

Simple Harmonic Motion (SHM) is characterized by a motion in which the restoring force is proportionate to the displacement of the body from its equilibrium position. Furthermore, the restoring force consistently acts toward the mean position.

SHM is a projection of uniform circular motion

Take into account a particle engaging in uniform circular motion with angular velocity ω in a clockwise direction, and its shadow undergoing a confined oscillatory motion in the horizontal plane. As we document different positions of both the particle and its shadow during the uniform circular motion (UCM), the following observations become apparent:





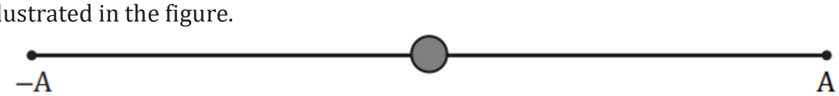
Upon the particle completing a full revolution along the circular path, its shadow (projection) shifts rightward, returns to its starting point, proceeds leftward, and once again returns to the initial position. This implies that when a particle undergoes uniform circular motion, the projection of this motion onto the diameter of the circle exhibits Simple Harmonic Motion (SHM). Consequently, SHM can be considered as the projection of uniform circular motion.

The following can be deduced:

1. The shadow (projection) undergoes oscillatory and periodic movements.
2. The shadow displays rectilinear motion, thus making the projection (shadow) of Uniform Circular Motion (UCM) a straight line. Consequently, Simple Harmonic Motion (SHM) is characterized by linear or straight-line motion.
3. The motion of both the particle and its shadow is periodic. Nevertheless, Uniform Circular Motion (UCM) itself does not qualify as an oscillatory motion; instead, its shadow exhibits oscillatory motion.

SHM is not only a physical observation but also a mathematical coincidence

Imagine a particle engaged in periodic oscillations, moving back and forth within the range of -A to A, as illustrated in the figure.



Let's suppose that the force exerted on the particle is,

$$F \propto -x$$

$$F = -kx \quad \text{... (1)}$$

The negative sign signifies that the force acts in a restoring manner. If m denotes the mass of the particle, equation (i) transforms into,

$$ma = -kx$$

$$a = -\left(\frac{k}{m}\right)x$$

$$a = -\omega^2 x$$

(Since k and m both are constants, we can define $\frac{k}{m} = \text{ Constant } = \omega^2$)

$$v\frac{dv}{dx} = -\omega^2 x$$

(Where v is the velocity of the particle)

$$\int vdv = -\int \omega^2 xdx$$

$$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + \frac{c^2}{2}$$

(Where $\frac{c^2}{2}$ is the constant of integration)

$$v^2 = c^2 - \omega^2 x^2$$
$$v = \sqrt{c^2 - \omega^2 x^2}$$
$$v = \omega\sqrt{\frac{c^2}{\omega^2} - x^2}$$
$$v = \omega\sqrt{A^2 - x^2}$$

$$\left[\begin{array}{l}\text{Since } \frac{c^2}{2} \text{ is a constant, } c^2 \text{ and } \omega^2 \text{ are also constants.} \\ \qquad \text{So, } \frac{c^2}{\omega^2} = A^2 = \text{ Constant}\end{array}\right]$$

Now, since $v = \frac{dx}{dt}$, we get,

$$\frac{dx}{dt} = \omega\sqrt{A^2 - x^2}$$
$$\int \frac{dx}{\sqrt{A^2 - x^2}} = \omega \int dt$$

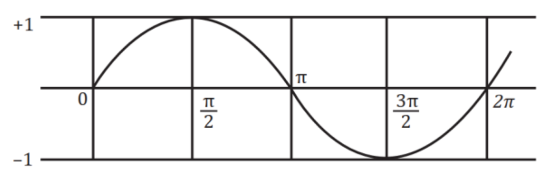
$\sin^{-1}\left(\frac{x}{A}\right) = \omega t + \phi$(Where ϕ is the integration constant)

$$\frac{x}{A} = \sin(\omega t + \phi)$$
$$x = A\sin(\omega t + \phi)$$

Hence, when the force acting on a particle follows the relationship $F \propto -x$ (indicating a linear and restoring force), the position of the particle changes according to $x = A.\ \sin(\omega t + \phi)$,Here, x represents the particle's displacement in relation to the mean position (O).

Properties of sine function

The sine function is both periodic and confined within a specific range, as depicted in the figure.



Given that the force acts in response to the relationship $F \propto -x$,

- The particle's displacement is expressed as $x = A, \sin(\omega t + \phi)$. Considering that the sine function ranges between +1 and −1, the displacement (x) of the particle is confined within the limits of (−A, A).
- The particle's velocity can be expressed as $v = \frac{dx}{dt} = A\omega\cos(\omega t + \phi)$ Considering that the cosine function has maximum and minimum values of +1 and −1, respectively, the velocity (v) of the particle will fall within the range of ($-A\omega$, $A\omega$).
- The particle's acceleration is characterized by $a = \frac{dv}{dt} = -A\omega^2\sin(\omega t + \phi)$ and the acceleration spans from ($-A\omega^2$, $A\omega^2$).

Thus, we observe that the position, velocity, and acceleration of the particle are confined within a specified range.

If a function f(x) has a time period T, then the function f (ax + b) will have a time period $\frac{T}{|a|}$ Therefore, given that any sine function, like sin(t), has a time period 2π, $\sin(\omega t + \phi)$ will have time period $\frac{2\pi}{\omega}$.

As a result, the position, velocity, and acceleration of the particle are confined within a range during its oscillatory motion with a time period.$\frac{2\pi}{\omega}$

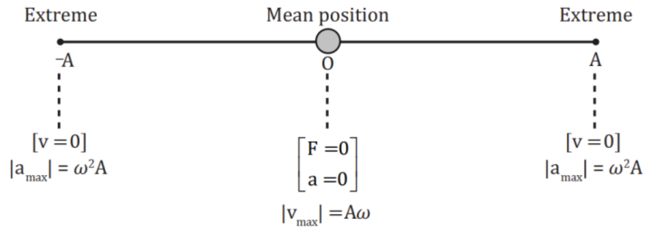
This movement exhibited by the particle is termed simple harmonic motion (SHM). The term SHM is derived from the sine function, which is a periodic and confined function recognized as a harmonic function, commonly referred to as a simple harmonic function.

The force ($F \propto -x$) is the cause of simple harmonic motion (SHM), while its periodicity and oscillatory motion are the outcomes of this force.

Properties of sine function

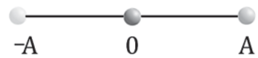
1. The force must be both linear and restoring.
2. The displacement (x) should change over time in accordance with a sine function.

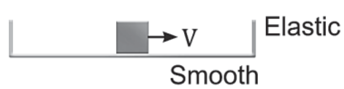
Key terms in SHM



1. **Mean Position:** As illustrated in the figure, point O is referred to as the mean position of the particle undergoing SHM.
    - ➢ The force (F) and acceleration (a) have magnitudes of zero at the mean position.
    - ➢ The velocity (v) possesses its maximum value, $|v_{max}| = A\omega$, at the mean position.
2. **Extreme position:** Points A and –A represent the two extreme positions, as depicted in the figure.
    - ➢ The velocity magnitude is zero at the extreme position.
    - ➢ The acceleration reaches its maximum value at the extreme position, and the maximum acceleration magnitude is $|a_{max}| = A\omega^2$.

Ex. Is the motion of a block on a frictionless surface confined between two elastic walls at each end considered simple harmonic motion (SHM)? Provide an answer along with an explanation.



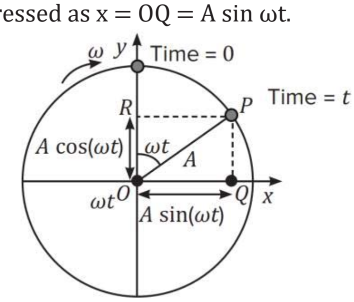


Sol. Although the block undergoes periodic and oscillatory motions, it cannot be categorized as simple harmonic motion (SHM) because the force acting on the block does not exhibit a restoring force pattern ($F \propto -x$).

SHM as a projection of Uniform Circular Motion

Imagine a particle engaged in uniform circular motion with angular velocity ω in a clockwise direction. Assume that at time t = 0, the particle is situated at the north pole of the circle, and at time t, it reaches position P, as depicted in the figure. Consequently, the latitude of the particle, or the angle covered at point P, is ωt.

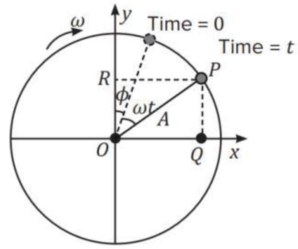
The projection of the particle is represented by points O and Q, corresponding to its positions at the north pole and point P, respectively. Considering the radius of the circle as A, it is evident that OR = A cos ωt and OQ = A sin ωt. Given that OQ aligns with the x-axis, the general displacement along the x-axis can be expressed as x = OQ = A sin ωt.



Initial phase or epoch (ϕ)

Assume that the particle does not commence its motion from the North Pole but instead starts at an initial angle ϕ with the y-axis, as illustrated in the figure. Suppose, at time t, the particle reaches point P, forming an angle of ωt from its initial position. Consequently, at time t, the particle is located at a latitude of $(\omega t + \phi)$.

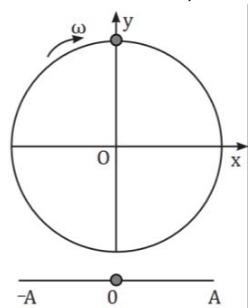
Consequently, the general displacement in this scenario is expressed as $x = OQ = A \sin(\omega t + \phi)$. Thus, the equation of the particle's displacement is obtained by projecting uniform circular motion, identical to the displacement of a particle undergoing Simple Harmonic Motion (SHM) under the influence of a force $F \propto -x$. It's worth noting that while the projection is considered along the x-axis in this instance, it could be carried out along any axis.



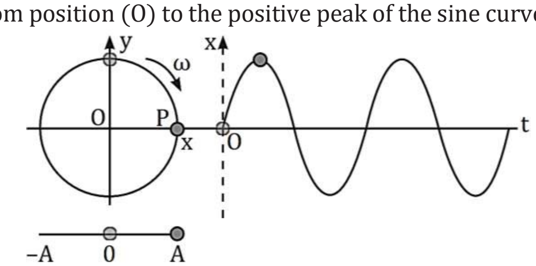
Note. Simple Harmonic Motion (SHM) can be conceptualized as the result of projecting uniform circular motion, and it is also the motion of a particle subjected to the force F ∝ – x. However, it is crucial to acknowledge that the sole cause of SHM is the linear motion of the particle under the influence of the force F ∝ – x.

Characteristics of SHM

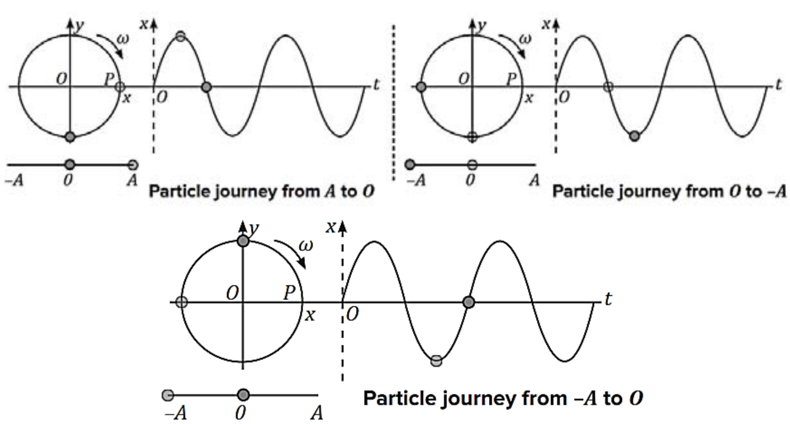
Take a particle undergoing Simple Harmonic Motion (SHM), as depicted in the figure. The particle's displacement is defined as $x = A \sin(\omega t + \phi)$. In the scenario where the particle commences Uniform Circular Motion (UCM) from the North Pole, where $\phi = 0$, the displacement simplifies to $x = A \sin \omega t$.



While the particle progresses from its initial position in Uniform Circular Motion (UCM) to point P, in the corresponding Simple Harmonic Motion (SHM), it transitions from the mean position (O) to the extreme end (A). In the time-variant diagram of the particle, this movement is illustrated as the particle shifting from position (O) to the positive peak of the sine curve, as depicted in the figure.



Likewise, the successive progression of its simple harmonic motion (SHM) from point A to O, O to –A, –A to O, along with the corresponding uniform circular motion (UCM) and the time-varying diagram, is illustrated below.



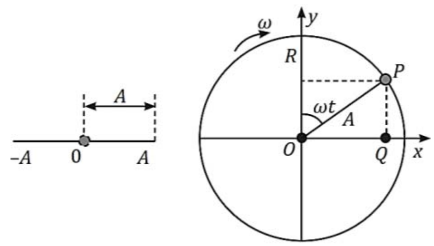
Particle journey from *A* to *O*

Particle journey from *O* to –*A*

Particle journey from –*A* to *O*

Amplitude (A):

The maximum magnitude of a particle's displacement from its mean position is referred to as its amplitude. Alternatively, one can state that the amplitude is the radius of the uniform circular motion (UCM) corresponding to the simple harmonic motion (SHM) of the particle.



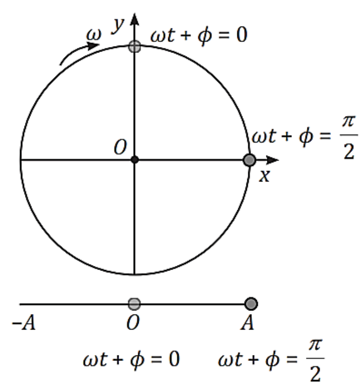
Time period (T):

The shortest time interval in which the motion of the particle undergoing simple harmonic motion (SHM) repeats itself is called the time period of the SHM. If the angular velocity of the particle is ω, then the time period of the motion is $T = \frac{2\pi}{\omega}$.

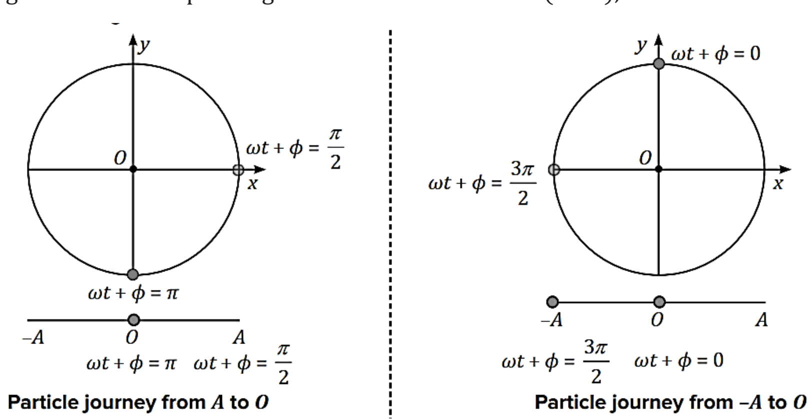
Phase angle (δ):

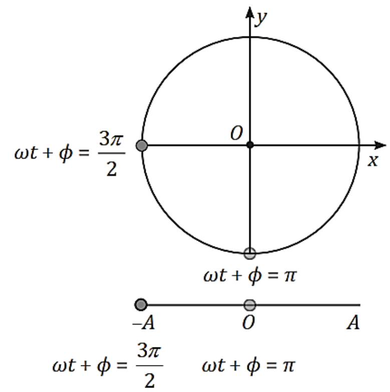
If the particle undergoing simple harmonic motion (SHM) has a displacement given by $x = A \sin(\omega t + \phi)$, then the expression $(\omega t + \phi)$ is referred to as the phase angle (δ) or the angle covered by the particle in uniform circular motion (UCM). In the context of SHM, this expression is termed the phase (δ) and it defines the position of the particle in the SHM.

Examining the quadrant analysis of uniform circular motion (UCM), when the particle transitions from the North Pole to the horizontal position, in its corresponding simple harmonic motion (SHM), the particle moves from the mean position O to one extreme end A, and the phase angle undergoes a change from 0 to. $\frac{\pi}{2}$, Respectively, as shown in the figure.



Likewise, its successive progression in simple harmonic motion (SHM) from A to O, O to –A, –A to O, along with the corresponding uniform circular motion (UCM), is illustrated in the figure below.





Particle journey from *O* to –*A*

Angular frequency (ω):

The rate of phase change (δ) in the simple harmonic motion (SHM) of a particle is referred to as the angular frequency of the SHM. It is defined as.$\omega = \frac{d\delta}{dt}$.

It is important to note that ω represents the angular frequency of simple harmonic motion (SHM), whereas it denotes the angular velocity of the uniform circular motion (UCM) of the particle.

Ex. The equation of a particle executing simple harmonic motion is $x = 5\sin(\pi t + \frac{\pi}{3})$m Write down the amplitude, time period, and maximum speed. Find the velocity at t = 1 s.

Sol. Given that the equation of the particle executing SHM is, $x = 5\sin(\pi t + \frac{\pi}{3})$m Comparing this equation with the general equation, $x = A\sin(\omega t + \phi)$, we get the following:

The amplitude of the particle is, A = 5 m

The angular frequency of the particle is, $\omega = \pi$ rad s^{-1}.

Therefore, the time period is, $T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2$ s

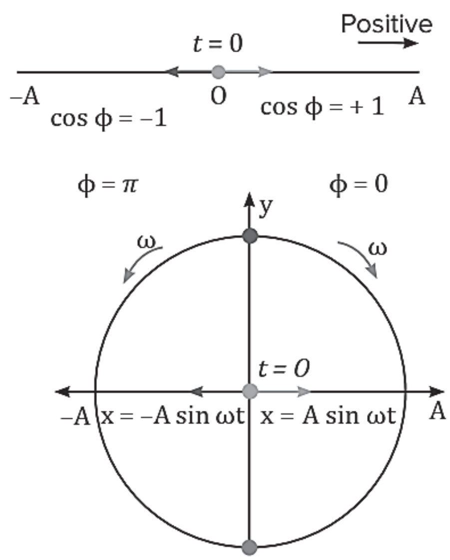
Since the displacement is defined as $x = A\sin(\omega t + \phi)$, the velocity is defined as $v = A\omega\cos(\omega t + \phi)$. Hence, the maximum velocity is, $v_{max} = A\omega = 5 \times \pi = 5\pi ms^{-1}$

From the given equation, the expression for velocity can be derived and that expression is given by, $v = A\omega\cos(\pi t + \frac{\pi}{3})ms^{-1}$ since the amplitude, A = 5 m and $\omega = \pi$ rad s^{-1}, the velocity for t = 1 s is,

$$v_{t=1} = 5\pi\cos((\pi \times 1) + \frac{\pi}{3})$$
$$v_{t=1} = 5\pi\cos(\frac{4\pi}{3})$$
$$v_{t=1} = -5\pi\cos(\frac{\pi}{3})$$
$$v_{t=1} = -\frac{5\pi}{2}ms^{-1}$$

Thus, the velocity of the particle at t = 1 s is, $-\frac{5\pi}{2}ms^{-1}$

Standard Equation of SHM



Particle starts from the mean position: In this scenario, at t = 0, the displacement of the particle is x = 0. The general equation for the displacement of a particle undergoing simple harmonic motion (SHM) is $x = A\sin(\omega t + \phi)$. By substituting t = 0 and x = 0 into the displacement equation, we obtain $0 = A\sin\phi$.

$\phi = 0$ or π [Since $\sin(0) = 0 = \sin(\pi)$ and A cannot be zero as it would make the particle motionless]

Two possible values of ϕ exist, depending on the context. Now, the formula for the velocity of the particle is $v = A\omega\cos(\omega t + \phi)$. At t = 0, the velocity expression becomes $v = A\omega\cos\phi$. Consequently, for $\phi = 0$, the velocity of the particle will be $v = A\omega$, and for $\phi = \pi$, the velocity of the particle will be $v = -A\omega$.

If we designate the right side of the mean position (O) as the positive side and the left side as the negative side, then the velocity of the particle will be positive while moving from O to A and negative while going from O to –A. Consequently, $\phi = 0$ for O to A and $\phi = \pi$ for O to –A, as depicted in the figure.

Thus, if the particle goes from O to A, the equation will be, $x = A\sin(\omega t + 0) = A\sin\omega t$ And if the particle goes from O to –A, the equation will be, $x = A\sin(\omega t + \pi) = -A\sin\omega t$ Therefore, in short, if the particle starts from the mean position, the equation will be, $x = \pm A\sin\omega t$.

Now, if we equate uniform circular motion (UCM) with the simple harmonic motion (SHM) of the particle, the corresponding equations will be $x = A\sin\omega t$ and $x = -A\sin\omega t$ for the clockwise and anti-clockwise rotations, respectively, as illustrated in the figure.

Particle starts from an extreme position

In this scenario, at $t = 0$, the velocity of the particle is $v = 0$, and the displacement of the particle is $x = \pm A$. Now, the general equation of displacement for a particle executing SHM is, $x = A\sin(\omega t + \phi)$. By putting $t = 0$ and $x = \pm A$ in the equation of displacement, we get, $\pm A = A\sin\phi$.

$$\phi = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

Thus, if the particle starts from + A, the equation will be,

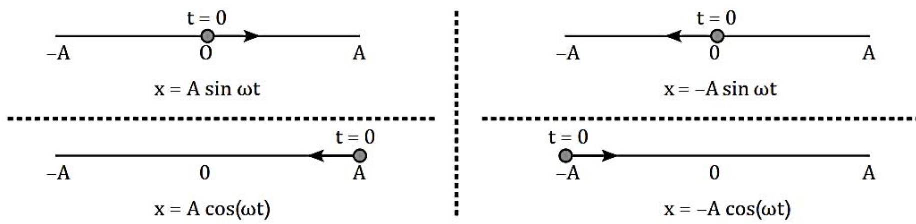
$$x = A\sin(\omega t + \frac{\pi}{2}) = A\cos\omega t$$

And if the particle starts from – A, the equation will be,

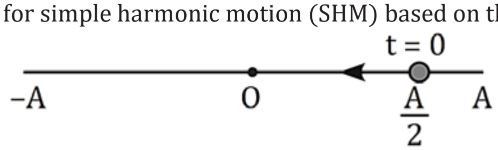
$$x = A\sin(\omega t + \frac{3\pi}{2}) = -A\cos\omega t$$

Therefore, in short, if the particle starts from the extreme position, the equation will be, $x = \pm A\cos\omega t$.

The summary of the standard equation of SHM is shown in the figure.



Ex. Express the equation for simple harmonic motion (SHM) based on the provided scenario.



Sol. Let the equation of SHM of the particle be $x = A\sin(\omega t + \phi)$. At $t = 0$, it is given that $x = +\frac{A}{2}$ Therefore, by putting this condition into the equation, we get.

$$+\frac{A}{2} = A\sin(\phi)$$

$$\sin(\phi) = \frac{1}{2}$$

$$\phi = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

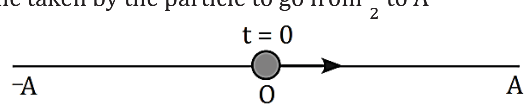
According to our choice, since the particle is pointing towards the left side here, the velocity of the particle would be negative. Now, the general expression of velocity is, $v = A\omega\cos(\omega t + \phi)$. Therefore, at $t = 0$, the velocity will be, $v = A\omega\cos\phi$ and $\cos\phi$ would be negative only for $\phi = \frac{5\pi}{6}$.

Hence, the equation of SHM of the given configuration will be, $x = A\sin(\omega t + \frac{5\pi}{6})$

Ex. A particle is performing SHM of amplitude A and time period T.

(a) Find the time taken by the particle to go from O to, $\frac{A}{2}$.

(b) Find the time taken by the particle to go from $\frac{A}{2}$ to A



Sol. Let the particle start from the mean position and hence, at $t = 0$, $x = 0$. Consider the equation of SHM to be, $x = A \sin \omega t$.

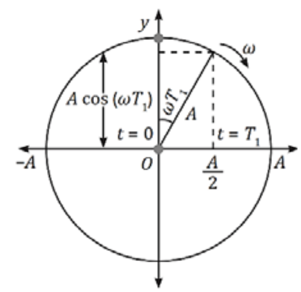
(a) Method 1:

At time $t = T_1$, let the displacement of the particle be $\frac{A}{2}$ Thus, by putting this condition into the equation of SHM, we get,

$$\frac{A}{2} = A\sin(\omega T_1)$$
$$\frac{1}{2} = \sin(\omega T_1)$$
$$\sin(\frac{\pi}{6}) = \sin(\omega T_1)$$
$$\omega T_1 = \frac{\pi}{6}$$
$$(\frac{2\pi}{T})T_1 = \frac{\pi}{6}$$
$$T_1 = \frac{T}{12}$$

Therefore, the time taken by the particle to go from O to $\frac{A}{2}$ is $\frac{T}{12}$.

Method 2:

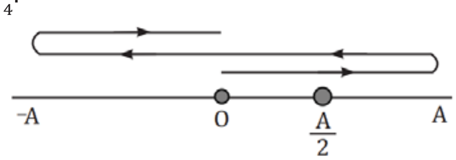


We know that if a particle executes a uniform circular motion, then its projection always executes a SHM along the diameter of the circle. At time $t = 0$, the particle is at the North Pole and its projection is on the mean position (O). Now, at time $t = T_1$, the particle which is executing the UCM is at latitude ωT_1 and its projection is at $x = \frac{A}{2}$ as shown in the given figure. If the radius of the circle is A, then the projection of the particle along the x-axis is, $x = A \sin \omega T_1$.

Therefore
$$\frac{A}{2} = A\sin(\omega T_1)$$
$$\frac{1}{2} = \sin(\omega T_1)$$
$$\sin(\frac{\pi}{6}) = \sin(\omega T_1)$$
$$\omega T_1 = \frac{\pi}{6}$$
$$(\frac{2\pi}{T})T_1 = \frac{\pi}{6}$$
$$T_1 = \frac{T}{12}$$

Therefore, Time taken by the particle to go from latitude O to ωT_1 = Time taken by the projection of the particle to go from O to $\frac{A}{2}$.

(b) For one complete rotation, the time period is T and the distance covered by the particle is 4A. Therefore, the time required to cover a distance of A is $\frac{T}{4}$ It means that to go from O to A, the time taken by the particle is $\frac{T}{4}$.



We have derived that the time taken by the particle to go from O to $\frac{A}{2}$ to $\frac{T}{12}$ Hence, the time taken by the particle to go from $\frac{A}{2}$ to A is $(\frac{T}{4} - \frac{T}{12}) = \frac{T}{6}$.

Note: Distance-wise, O to $\frac{A}{2}$ and $\frac{A}{2}$ to A are equal, but their time intervals are not equal because the velocities of the particle at O and $\frac{A}{2}$ are different.

Note: In SHM, the smallest indivisible unit of distance is A and the smallest indivisible unit of time is $\frac{T}{4}$.