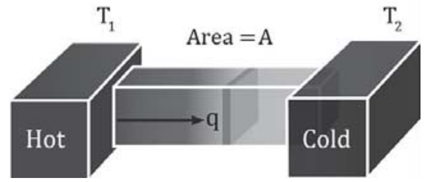
CONDUCTIVITY AND CONVECTION

Steady State



It is a condition where the flow of heat energy through a specific medium remains steady.

In this scenario, where $T_1 > T_2$), heat flows from left to right, and q denotes the rate of heat flow. Throughout the heat transfer process, the temperature at each point evolves over time. Heat flow persists until both bodies reach a uniform temperature.

This state is referred to as thermal equilibrium. In this stable condition, the temperature at any given point within the medium remains constant over time. This signifies that the rate of temperature change at any point within the medium is zero. Consequently, the heat arriving at that point is effectively transferred to the next point in succession.

In a steady state, the temperature of a specific cross-sectional area within the medium remains constant over time. This is in accordance with Fourier's law of heat conduction, which defines the rate of heat transfer.

Rate of heat transfer $q = \frac{Q}{t} = -kA\frac{dT}{dx}$

Thermal conductivity $k = -\frac{q}{A\frac{dT}{dx}}$

Thermal resistance $R_{Th} = \frac{L}{kA}$

Combination of metallic slabs

Analogy between thermal resistance and electrical resistance

We are aware that the electric current is expressed as:

$$i = \frac{q}{t} = \frac{\Delta V}{R} \text{ and, } R = \rho\frac{L}{A} \quad \dots (1)$$

The thermal current is given as follows:

$$i_T = \frac{q}{t} = kA(\frac{\Delta T}{L}) = \frac{\Delta T}{R_T} \quad \dots (2)$$

Where $R_T = \frac{1}{k}(\frac{L}{A})$

In the above equation,

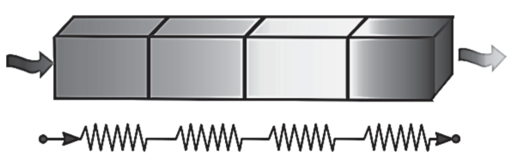
ρ = Electrical resistivity k = Thermal conductivity of the material

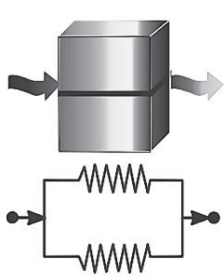
R = Electrical resistance RT = Thermal resistance

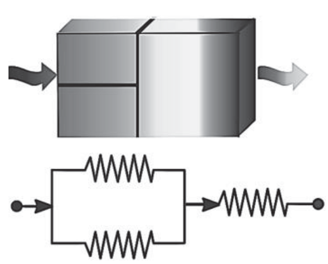
By comparing equations (1) and (2), it becomes evident that R and R_T are similar terms.

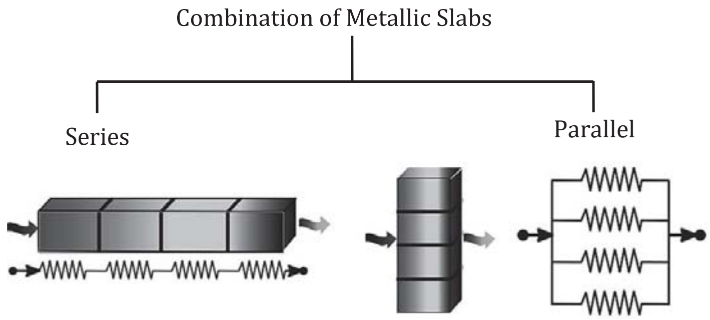
Hence, we can depict the resistance presented by a metal slab to the flow of heat through it using electrical resistances.

The diagram below illustrates a combination of metallic slabs alongside their counterparts represented by a combination of resistors.



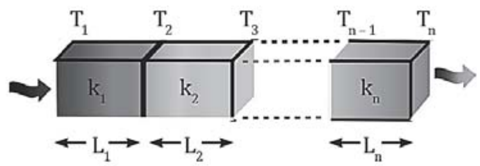






Series combination

In a series arrangement of slabs, there are n slabs with respective lengths L_1, L_2... Ln and thermal conductivities k_1, k_2... kn. These slabs are connected in series, as depicted in the figure.



Heat is provided to the initial slab, and the temperatures at the interfaces are T_1, T_2... T_n – 1, T_n (where $T_1 > T_n$). The heat flow remains consistent across all the slabs.

Since the heat current, q, remains constant, we can write it as follows:

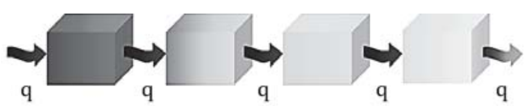
$$q_1 = q_2 = q_3 = q_n = q = \frac{T_1 - T_n}{R_{eq}}$$

$$q = \frac{\Delta T}{R_{eq}}$$

$$R_{eq} = \frac{T_1 - T_n}{q}$$

$$\frac{T_{n-1} - T_n}{q} + \frac{T_{n-2} - T_{n-1}}{q} + \cdots . + \frac{T_1 - T_2}{q}$$

$$R_{eq} = R_1 + R_2 + \cdots + R_n$$



This formula mirrors the series combination of resistors. Therefore, the equivalent thermal resistance is determined by.

$$R_{eq} = \frac{L}{k_{eq}A} = \frac{L_1}{k_1 A_1} + \frac{L_2}{k_2 A_2} + \cdots . + \frac{L_n}{k_n A_n}$$

$$\frac{L_1 + L_2 + \cdots .. + L_n}{k_{eq}A} = \frac{L_1}{k_1 A_1} + \frac{L_2}{k_2 A_2} + \cdots + \frac{L_n}{k_n A_n}$$

If the cross-sectional areas of all the slabs are identical ($A_1 = A_2 = \ldots = A_n$), then the equivalent thermal conductivity is expressed as follows:

$$k_{eq} = \frac{L_1 + L_2 + \cdots + L_n}{\frac{L_1}{k_1} + \frac{L_2}{k_2} + \cdots + \frac{L_n}{k_n}}$$

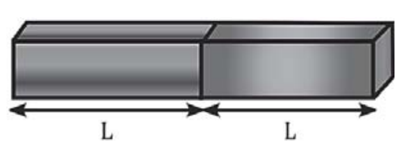
Now, if the length of all the slabs are equal, then,

$$k_{eq} = \frac{n}{\frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}}$$

For the combination of two slabs, the same formula gets modified as follows:

$$k_{eq} = \frac{2k_1 k_2}{k_1 + k_2}$$

Ex. Two metallic slabs of the same area of cross section, length, and thickness having thermal conductivities 300 $Wm^{-1}°C^{-1}$ and 200 $Wm^{-1}°C^{-1}$, respectively, are connected in a series combination. Calculate the equivalent thermal conductivity in terms of $Wm^{-1}°C^{-1}$.



(a) 240 (b) 360 (c) 250 (d) 275

Sol. Given,

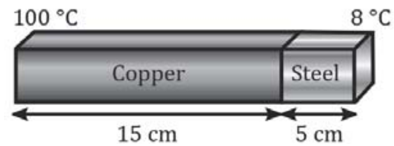
$$A_1 = A_2 = A$$
$$L_1 = L_2 = L$$
$$k_1 = 300 Wm^{-1}\,°C^{-1}$$
$$k_2 = 200 Wm^{-1}\,°C^{-1}$$

By using the equivalent thermal conductivity equation, we get the following.

$$k_{eq} = \frac{2k_1k_2}{k_1+k_2} = \frac{2\times300\times200}{300+200} = \frac{120000}{500} = 240 Wm^{-1}\,°C^{-1}$$

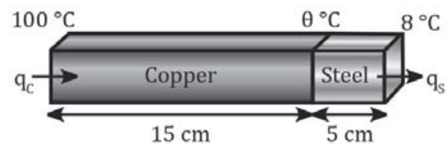
Thus, option (A) is the correct answer.

Ex. The coefficient of thermal conductivity of copper is nine times than that of steel. In the composite slab shown in the figure, what will be the temperature at the junction?



(a) 77 °C (b) 75 °C
(c) 67 °C (d) 27 °C

Sol. Given, $k_C = 9k_S$



k_C = Thermal conductivity of copper
k_S = Thermal conductivity of steel
q_C = Rate of heat transfer for copper
q_S = Rate of heat transfer for composite steel
$\theta = ?$

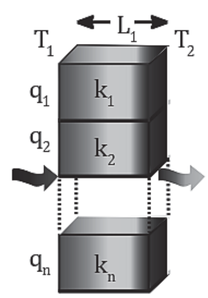
By using the concept of the rate of heat transfer,

$$q_C = q_S$$
$$k_C = 9k_S$$
$$\frac{k_C(100-\theta)}{15} = \frac{k_S(\theta-8)}{5}$$
$$\frac{9k_S(100-\theta)}{15} = \frac{k_S(\theta-8)}{5}$$
$$\frac{9(100-\theta)}{15} = \frac{(\theta-8)}{5}$$
$$3(100-\theta) = \theta - 8$$
$$\theta = \frac{308}{4} = 77°C$$

Thus, option (A) is the correct answer.

Parallel Combination

In a parallel arrangement of slabs, the temperature change (ΔT) remains constant across all the slabs. However, the heat flow varies among the different slabs.



The total heat flow for the parallel arrangement of the slabs is expressed as.

$$q_{net} = q_1 + q_2 + \cdots + q_n$$
$$q_{net} = \frac{(T_1-T_2)}{R_1} + \frac{(T_1-T_2)}{R_2} + \cdots + \frac{(T_1-T_2)}{R_n}$$
$$\frac{(T_1-T_2)}{R_{eq}} = \frac{(T_1-T_2)}{R_1} + \frac{(T_1-T_2)}{R_2} + \cdots + \frac{(T_1-T_2)}{R_n}$$
$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}$$

The net thermal conductivity (k_{eq}) is given by,

$$q_{net} = q_1 + q_2 + \cdots + q_n$$
$$q_{net} = k_1A_1\frac{(T_1-T_2)}{L} + k_2A_2\frac{(T_1-T_2)}{L} + \cdots + k_nA_n\frac{(T_1-T_2)}{L}$$

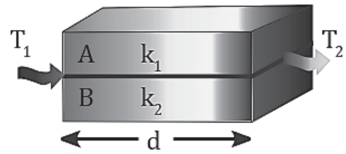
Also,

$$q_{net} = k_{eq}\frac{(A_1+A_2+\cdots+A_n)(T_1-T_2)}{L}$$
$$k_{eq} = \frac{k_1A_1+k_2A_2+\cdots+k_nA_n}{A_1+A_2+\cdots+A_n}$$

For the slabs having an equal area, we get the following:

$$A_1 = A_2 = \cdots = A_n$$
$$k_{eq} = \frac{k_1+k_2+\cdots+k_n}{n}$$

Ex. Two dimensionally similar slabs A and B of different materials are welded together, as shown in the figure, and their thermal conductivities are k_1 and k_2. What will be the thermal conductivity of the composite slab?



(a) $\frac{(k_1+k_2)}{2}$ (b) $\frac{3(k_1+k_2)}{2}$

(c) $(k_1 + k_2)$ (d) $2(k_1 + k_2)$

Sol. Since both the slabs are identical in terms of dimensions, we get the following:

$$k_{eq} = \frac{k_1+k_2+\cdots+k_n}{n}$$

$$k_{eq} = \frac{k_1+k_2}{2}$$

Thus, option (A) is the correct answer.