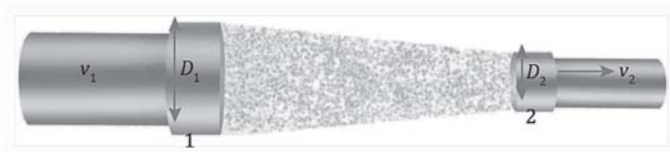
BERNOULLI'S EQUATION

Equation of Continuity:

The equation of continuity stems from the principle of mass conservation.



As per the principle of mass conservation, the mass of a fluid passing through a tube with a non-uniform cross-section remains constant over time. In the scenario depicted in the preceding figure, the mass flowing through cross-section 1 equals that flowing through cross-section 2.

$$(\text{Mass flow rate})_1 = (\text{Mass flow rate})_2$$

$$A_1 v_1 \rho_1 = A_2 v_2 \rho_2.$$

In the case of a fluid with constant density,

$$\rho_1 = \rho_2$$

$$\Rightarrow \quad A_1 v_1 = A_2 v_2$$

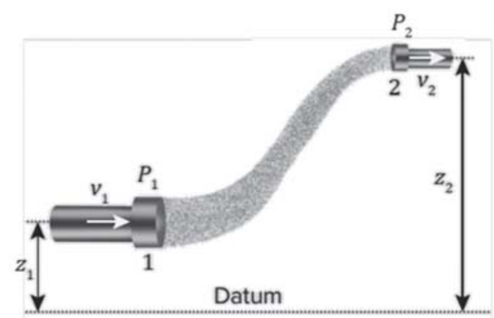
Bernoulli’s Principle:

For a continuous, incompressible, and non-viscous fluid flow, the total energy per unit volume remains unchanged. This means that the combined energy per unit volume—pressure energy, kinetic energy, and potential energy—remains constant. In mathematical terms,
this is expressed as:

$$P + \frac{1}{2}\rho v^2 + \rho g h = \text{ Constant}$$

Bernoulli's Equation:

Imagine a steady flow of a non-viscous, incompressible fluid through a pipe of varying cross-sections. The ends of the pipe are at different elevations relative to a reference point (or datum), as depicted in the illustration. Let's examine a thin layer of the fluid transitioning from section 1 to section 2 at heights z_1 and z_2 above the datum, as illustrated.



Let P_1, A_1, v_1, and P_2, A_2, v_2 be the pressure, area, and velocity at section 1 and section 2, respectively

Now, it is understood that in a tube with cross-sectional area A, when a pressure force P moves the liquid through a distance Δx, the work performed by the pressure force = the force × by the displacement,

which is expressed as $\quad PA\Delta x = PV$,

where V represents the volume of the thin layer.

The work done by the pressure force at section 1 = 1 = $P_1 A_1 \Delta x_1 = P_1 V$

The work done by the pressure force at section 2 = $-P_2 A_2 \Delta x_2 = -P_2 V$

(The negative sign denotes that the displacement (Δx_2) and the force ($P_2 A_2$) acting on the thin layer at section 2 are in opposite directions.)

Now,

The overall work done by the pressure force,

$$W = P_1 V - P_2 V$$

Now, in accordance with the law of conservation of energy, this work should equate to the alteration in the kinetic and potential energies of the liquid transitioning from section 1 to section 2.

$$\therefore \quad P_1 V - P_2 V = \Delta KE + \Delta PE.$$

$$\Rightarrow \quad (P_1 - P_2)V = \frac{1}{2}m(v_2^2 - v_1^2) + mg(z_2 - z_1)$$

$$\Rightarrow \quad (P_1 - P_2) = \frac{1}{2}\left(\frac{m}{V}\right)(v_2^2 - v_1^2) + \left(\frac{m}{V}\right)g(z_2 - z_1)$$

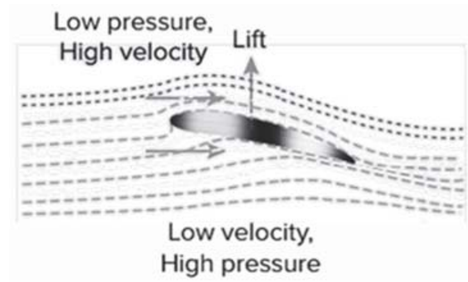
$$\Rightarrow \quad (P_1 - P_2) = \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(z_2 - z_1)$$

$$\Rightarrow \quad P_1 + \frac{1}{2}\rho v_1^2 + \rho g z_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g z_2$$

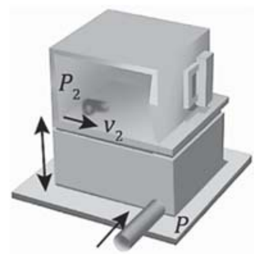
$$\Rightarrow \quad \mathrm{P} + \frac{1}{2}\rho \mathrm{v}^2 + \rho \mathrm{g z} = \text{Constant}$$

Applications of Bernoulli's Principle:

Because of the wing's shape, often referred to as an airfoil, the airflow around an airplane is divided at the wing's leading edge. The air travels above and below the wing at varying velocities, resulting in a pressure disparity that ensures it reaches the trailing edge of the wing simultaneously. This phenomenon contributes to generating lift.



Ex. Water is supplied to a house through a pipe with an inlet diameter of 2 cm and an absolute pressure of 4×10^5 Pa. A pipe with a diameter of 1 cm directs water to the bathroom on the second floor, which is situated 5 m above. Given that the flow speed at the inlet pipe is 1.5 ms^{-1}, determine the flow speed in ms^{-1} and the pressure in Pa in the bathroom.



Sol. Given,

$$\mathrm{D_1} = 2\ \mathrm{cm}$$

$$\mathrm{P_1} = 4 \times 10^5\ \mathrm{Pa}$$

$$\mathrm{D_2} = 1\ \mathrm{cm}$$

$$\Delta \mathrm{h} = 5\ \mathrm{m}$$

$$\mathrm{v_1} = 1.5\ \mathrm{ms^{-1}}$$

$$\mathrm{P_2} = ?$$

$$\mathrm{v_2} = ?$$

Utilizing the equation of continuity at points 1 and 2 yields the following:

$$\mathrm{A_1 v_1} = \mathrm{A_2 v_2}$$

$$\Rightarrow \quad \frac{\pi}{4} \times D_1^2 \times v_1 = \frac{\pi}{4} \times D_2^2 \times v_2$$

$$\Rightarrow \quad \mathrm{v_2} = \frac{\mathrm{D_1^2 \times v_1}}{\mathrm{D_2^2}}$$

$$\Rightarrow \quad \mathrm{v_2} = \frac{(2\times10^{-2})^2 \times 1.5}{(1\times10^{-2})^2}$$

$$\Rightarrow \quad \mathrm{v_2} = 6\ \mathrm{ms^{-1}}$$

When applying Bernoulli's equation at inlet 1 and outlet 2,

$$(P_1 - P_2) = \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(h_2 - h_1)$$

$$\Rightarrow \quad (4 \times 10^5) - P_2 = (\frac{1}{2} \times 1000 \times (6^2 - 1.5^2)) + (1000 \times 10 \times 5)$$

$$\Rightarrow \quad \mathrm{P_2} = (4 \times 10^5) - 16875 - 50000$$

$$\Rightarrow \quad \mathrm{P_2} = (3.3 \times 10^5)\mathrm{Nm^{-2}}$$

$$\Rightarrow \quad 6, 3.3 \times 10^5$$