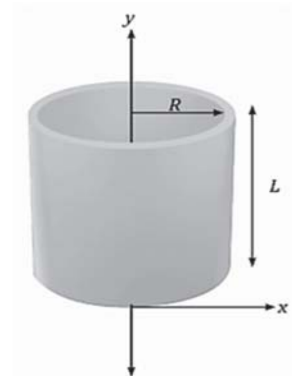
MOMENT OF INERTIA OF 3D BODIES AND PARALLEL AXIS THEOREM

Moment of Inertia of: Hollow Cylinder, Solid Cylinder, Hollow Sphere, Solid Sphere and Rectangular Lamina:

Moment of Inertia of a Thin Uniform Hollow Cylinder:

The calculation of the moment of inertia for a slender, even-walled hollow cylinder, characterized by its mass MM, length LL, and radius RR, is delineated in the subsequent paragraphs.

Given that the cylinder's mass is uniformly spread across its surface area, we ascertain the mass per unit area as follows:



$$\sigma = \frac{M}{2\pi RL}$$

We can conceptualize the slender hollow cylinder as comprising a series of closely stacked thin rings. To analyze its structure, we establish the y-axis as the centroidal axis, with the center of the base serving as the origin. Now, we focus on a specific ring, characterized by its thickness dy and mass dmdm, situated y units above the origin. So,

$$\mathrm{dm} = \sigma(2\pi \mathrm{R})\mathrm{dy}$$

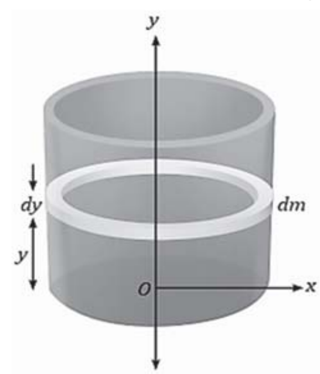
The calculation of the moment of inertia for this specific segment concerning the centroidal axis entails further examination.

$$\mathrm{dI} = \mathrm{dmR}^2$$

$$\mathrm{dI} = (\sigma(2\pi \mathrm{R})\mathrm{dy})\mathrm{R}^2$$

$$\mathrm{dI} = \sigma(2\pi \mathrm{R}^3)\mathrm{dy}$$

Consequently, the moment of inertia for the entire cylinder can be determined through comprehensive analysis.



$$\int_0^l dI = \int_0^L \sigma(2\pi R^3)dy$$

$$\int_0^l dI = 2\pi\sigma R^3 \int_0^L dy$$

$$[I]_0^l = 2\pi\sigma R^3 [y]_0^L$$

Our understanding encompasses the fact that,

$$\sigma = \frac{M}{2\pi RL}$$

$$\mathrm{I} = (2\pi \mathrm{R}^3)\left(\frac{\mathrm{M}}{2\pi \mathrm{RL}}\right)\mathrm{L}$$

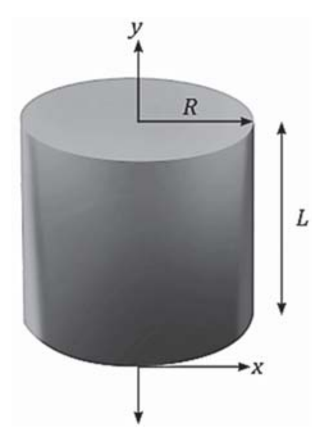
$$\mathrm{I} = \mathrm{MR}^2$$

Therefore, the moment of inertia pertaining to a uniform hollow cylinder with respect to its centroidal axis can be derived as follows:

$$\mathrm{I} = \mathrm{MR}^2$$

Moment of Inertia of a Uniform Solid Cylinder:

The subsequent lines outline the procedure for calculating the moment of inertia of a uniform solid cylinder characterized by mass M, length L, and radius R concerning its centroidal axis, which is parallel to the height of the cylinder. To comprehend its structure, we envision the solid cylinder as comprising successive thin discs stacked precisely on top of one another.

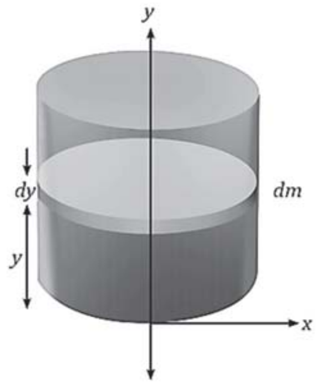


For analysis purposes, we establish the y-axis as the centroidal axis, with the center of the base serving as the origin. Now, we direct our attention to a particular disc, characterized by its thickness dy and possessing mass dmdm, situated at a distance y above the origin.

$$dm = \rho(\pi R^2)dy$$

The inquiry pertains to the determination of the moment of inertia for this particular segment concerning its centroidal axis.

$$dI = \frac{1}{2}dmR^2$$
$$dI = \frac{1}{2}(\rho(\pi R^2)dy)R^2$$
$$dI = \frac{1}{2}(\pi\rho R^4)dy$$



The expression refers to the moment of inertia encompassing the entirety of the cylinder.

$$\int_0^I dI = \int_0^L \frac{1}{2}(\pi\rho R^4)dy$$
$$\int_0^I dI = \frac{1}{2}(\pi\rho R^4)\int_0^L dy$$
$$[I]_0^I = \frac{1}{2}(\pi\rho R^4)[y]_0^L$$

We know that, $\rho = \frac{M}{\pi R^2 L}$

$$I = \frac{1}{2}(\pi R^4)(\frac{M}{\pi R^2 L})L$$
$$I = \frac{MR^2}{2}$$

Therefore, the forthcoming statement elucidates the moment of inertia concerning a uniform solid cylinder with respect to its centroidal axis.

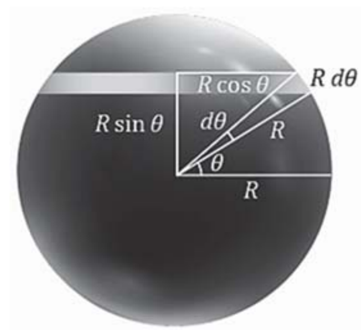
$$I = \frac{MR^2}{2}$$

Moment of Inertia of a Uniform Hollow Sphere:

The forthcoming lines outline the procedure for calculating the moment of inertia of a uniform hollow sphere characterized by mass MM and radius RR concerning its centroidal axis.

The entire mass of the sphere is uniformly distributed across its surface area, thus yielding the mass per unit area.

$$\sigma = \frac{M}{4\pi R^2}$$



The structure of the hollow sphere suggests a composition of successive thin rings, each with different radii, carefully arranged in layers.

To facilitate analysis, we designate the y-axis as the centroidal axis, locating the origin at the sphere's center. Let's focus on a particular ring, characterized by its mass dmdm as illustrated.

This chosen segment is positioned at an angle θ relative to the x-axis, and the ring itself spans an angle of dθ at the origin.

Therefore, the radius of the selected ring corresponds to Rcosθ.

When the ring is sectioned vertically and unfolded, it transforms into a rectangle with a length of 2πRcosθ and a width of $Rd\theta$.

Consequently, the area enclosed by the ring can be expressed as dA=(2πRcosθ) Rdθ.

Subsequently, the mass of the ring is given by dm=σdA,

where σ represents the mass per unit area. Substituting dA into the expression yields

$$dm=\sigma(2\pi R\cos\theta)\,Rd\theta.$$

Finally, the moment of inertia concerning this segment with respect to the centroidal axis is determined.

$$dI = dm(R\cos\,\theta)^2$$
$$dI = (\sigma(2\pi R\cos\,\theta)Rd\theta)(R\cos\,\theta)^2$$
$$dI = (2\pi\sigma R^4)\cos^3\,\theta d\theta$$

The moment of inertia of the entire sphere is,

$$\int_0^l dI = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}(2\pi\sigma R^4)\cos^3\theta d\theta$$
$$\int_0^l dl = (2\pi\sigma R^4)\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}(\frac{\cos\,3\theta+3c\quad\theta}{4})\,d\theta$$
$$(\because \cos\,3\theta = 4\cos^3\,\theta - 3\cos\,\theta)$$

$$[I]_0^I = \frac{\pi\sigma R^4}{2}\left[\frac{\sin\ 3\theta}{3} + 3\sin\ \theta\right]_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}$$
$$I = \frac{\pi\sigma R^4}{2}\left(\left(-\frac{1}{3}+3\right)-\left(\frac{1}{3}-3\right)\right)$$
$$I = \frac{\pi\sigma R^4}{2}\left(\frac{16}{3}\right)$$

It is within our knowledge that,

$$\sigma = \frac{M}{4\pi R^2}$$
$$I = \left(\frac{M}{4\pi R^2}\right)\left(\frac{\pi R^4}{2}\right)\left(\frac{16}{3}\right)$$
$$I = \frac{2}{3}MR^2$$

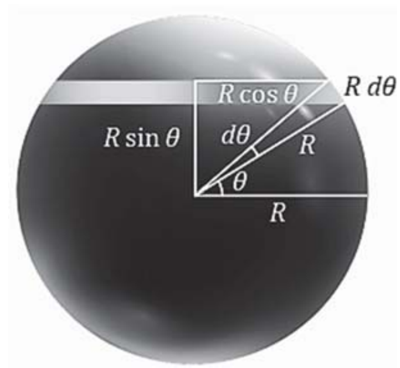
Therefore, the moment of inertia concerning a uniform hollow sphere with respect to its centroidal axis is as follows:

$$I = \frac{2}{3}MR^2$$

Solid Sphere and Rectangular Lamina:

Moment of Inertia of a Uniform Solid Sphere:

The forthcoming lines detail the computation of the moment of inertia for a uniform solid sphere with mass MM and radius RR regarding its centroidal axis.



The entire mass of the sphere is uniformly spread across its entire volume, thus yielding the mass per unit volume.

$$\rho = \frac{3M}{4\pi R^3}$$

Now, let's consider a specific disc of mass dmdm, as depicted in the figure. This selected segment is positioned at an angle θ with respect to the x-axis, and the disc itself encompasses an angle dθ at the origin.

Consequently, the radius of the chosen disc is Rcosθ, and its thickness is Rcosθdθ.

Therefore, the volume of the disc can be expressed as

$$dV=(\pi R^2\cos^2\theta)R\cos\theta d\theta.$$

Hence, the mass of the disc is $dm=\rho\ dV$dm=ρdV, where ρ denotes the mass per unit volume.

Thus, we obtain $dm=\rho(\pi R^3\cos^3\theta)d\theta$.

$$dI = \frac{1}{2}dm(R\cos\ \theta)^2$$
$$dI = \frac{1}{2}(\rho(\pi R^3\cos^3\ \theta)d\theta)R^2\cos^2\ \theta$$
$$dI = \frac{1}{2}(\rho\pi R^5)\cos^5\ \theta d\theta$$

The calculation pertains to the moment of inertia encompassing the entirety of the sphere.

$$\int_0^I dI = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{1}{2}(\rho\pi R^5)\cos^5\ \theta d\theta$$
$$\int_0^I dI = \frac{1}{2}(\rho\pi R^5)\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\cos^5\ \theta d\theta$$
$$I = \left(\frac{\rho\pi R^5}{2}\right)\left(\frac{16}{15}\right)$$

It is understood that, $\rho = \frac{3M}{4\pi R^3}$

$$I = \left(\frac{3M}{4\pi R^3}\right)\left(\frac{\pi R^5}{2}\right)\left(\frac{16}{15}\right)$$
$$I = \frac{2}{5}MR^2$$

Therefore, the moment of inertia for a uniform solid sphere concerning its centroidal axis is as follows:

$$I = \frac{2}{5}MR^2$$

Moment of Inertia of a Uniform Rectangular Lamina:

The procedure for calculating the moment of inertia for a uniform rectangular lamina with mass M, concerning the centroidal axis perpendicular to the plane of the lamina, is detailed in the following page.

Considering the three mutually perpendicular axes denoted by x, y, and z as illustrated in the figure, the moment of inertia of the rectangular lamina about the centroidal axis is denoted as I_Z=I.

The moment of inertia about the x-axis is $I_x = \frac{Mb^2}{12}$ and about the y-axis is $I_y = \frac{Mb^2}{12}$.

According to the perpendicular axis theorem, the total moment of inertia I equals the sum of the individual moments of inertia along the x and y axes, expressed as $I = I_x + I_y$.

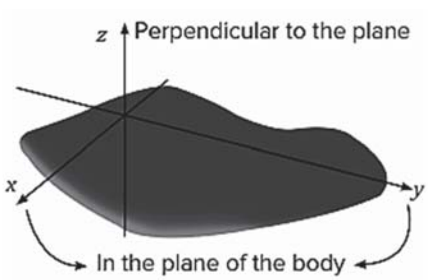
$$I = \frac{Ml^2 + Mb^2}{12} = \frac{M(l^2 + b^2)}{12}$$

Therefore, the moment of inertia concerning a uniform rectangular lamina about its centroidal axis perpendicular to its plane is as follows:

$$I = \frac{M(l^2 + b^2)}{12}$$

Perpendicular Axis Theorem:

The concept known as the perpendicular axis theorem establishes that when considering the moment of inertia of a planar object, denoted as z-axis, which is perpendicular to the plane of the object and intersects with two mutually perpendicular axes (x and y) located within the object's plane, the resulting moment of inertia is equal to the combined sum of the object's moments of inertia about the two perpendicular axes within the object's plane.



Proof:

Let's consider a particle belonging to the planar sheet, denoted by mi, positioned at coordinates (x_i, y_i). The distance of this particle from the z-axis is $(r_i = x_i^2 + y_i^2)^{205}$.

Therefore, the moment of inertia of this particle about the z-axis is given by

$$dI_z = m_i r_i^2 = m_i(x_i^2 + y_i^2).$$

Additionally, y_i and x_i represent the perpendicular distances from the y and x axes, respectively.

The moment of inertia of the particle about the x-axis is $dI_x = m_i y_i^2$

while the moment of inertia about the y-axis is $dI_y = m_i x_i^2$.

So,

$$dI_z = m_i(x_i^2 + y_i^2) = m_i x_i^2 + m_i y_i^2$$

$$dI_z = dI_y + dI_x$$

By integrating this equation, we arrive at

$$I_z = I_y + I_x$$