DYNAMICS OF PURE ROTATIONAL MOTION

In pure rotational motion, the axis of rotation remains unchanging. Here are the key formulas essential for analyzing pure rotational motion:

$$\vec{\tau}_{\text{hinge}} = I_{\text{hinge}}\,\vec{\alpha} \;\ldots\ldots\ldots (i)$$

$$\sum(\vec{F}_{\text{ext}})_{\text{sys}} = m\vec{a}_{\text{com}} \;\ldots\ldots\ldots (ii)$$

Equation (ii) can be expressed in a more detailed manner by breaking it down into its component form as follows:

$$\sum\vec{F}_t = m\vec{a}_t$$

And,

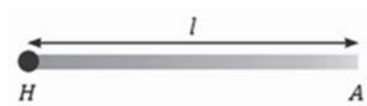
$$\sum\vec{F}_r = m\vec{a}_r$$

Here, $|a_t| = \alpha r_{com}$ and, $|a_r| = \omega^2 r_{com}$

The total kinetic energy equals the rotational kinetic energy,

which is $= \frac{1}{2}\mathrm{I}_{\text{hinge}}\,\omega^2 \;\ldots\ldots\ldots (iii)$

Ex. A uniform rod with mass MM and length LL has the ability to rotate within a vertical plane, pivoting about a smooth horizontal axis located at point H.



(a) Determine the total hinge force when the rod is positioned horizontally.

(b) Determine the angular velocity, acceleration, and the total hinge force when the rod reaches a vertical orientation.

(a) Calculating the total hinge force when the rod is in a horizontal position: As the rod is released from a state of rest, the initial angular velocity is zero ($\omega = 0$). When the rod is released, the weight (Mg) acts at the midpoint ($\frac{\mathrm{L}}{2}$ length) of the rod.

The torque around the hinge point can be expressed as follows:

$$\tau_H = I_H\alpha$$

$$\Rightarrow \quad (Mg)\left(\frac{L}{2}\right) = \left(\frac{ML^2}{3}\right)\alpha$$

[$\because$ The moment of inertia of the rod about one end is given by $\frac{ML^2}{3}$]

$$\Rightarrow \quad \alpha = \frac{3g}{2L} \;\ldots\ldots\ldots\ldots (i)$$

While the rod undergoes vertical circular motion, it experiences both radial and tangential acceleration. Given that this motion occurs within the vertical plane, the hinge forces within this plane could manifest in any direction. Due to the uncertainty regarding the resultant force's direction, we opt to decompose it into components (N_1 and N_2) aligned with the radial and tangential acceleration directions. In the tangential direction,

$$\mathrm{F_t = ma_t}$$

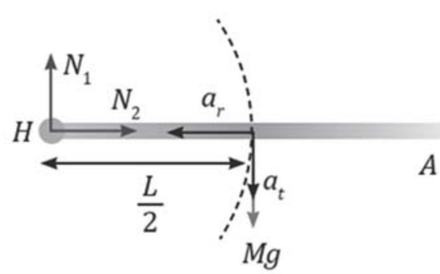
$$\Rightarrow \quad \mathrm{Mg - N_1 = M\left(\alpha\frac{L}{2}\right)}$$

By substituting the value of αα from equation (i), we obtain the following:

$$\Rightarrow \quad \mathrm{Mg - N_1 = M\left(\frac{3g}{2L}\right)\left(\frac{L}{2}\right)}$$

$$\Rightarrow \quad \mathrm{Mg - N_1 = M\left(\frac{3g}{4}\right)}$$

$$\Rightarrow \quad \mathrm{N_1 = \frac{Mg}{4}}$$



Likewise,
In the radial direction,

$$F_r = ma_r$$

$$-N_2 = M(\omega^2 \frac{L}{2}) = 0$$

When ω=0, we obtain the following: $N_2 = 0$

The total hinge force is determined as follows:

$$N = \sqrt{N_1^2 + N_2^2} = \frac{Mg}{4}$$

[$\because$ $N_2 = 0$]

As N_1 is directed upwards, the resultant hinge force is likewise directed upwards.

(b) Determining the angular velocity, acceleration, and the total hinge force when the rod reaches a vertical position. When the rod becomes vertical, the weight force aligns with the hinge. Hence, the torque about the hinge at this juncture can be expressed as follows:

$$\Sigma\vec{\tau}_H = I_H\vec{\alpha}$$

$\Rightarrow$ $$Mg(0) = \frac{ML^2}{3}(\alpha)$$

[$\because$ The rod's moment of inertia about one end is: $\frac{ML^2}{3}$]

$\Rightarrow$ $$\alpha = 0$$

Now, let's take the initial horizontal position as the reference for the initial potential energy. Since the rod was initially at rest, both its kinetic energy and potential energy are zero in this state. By applying the principle of conservation of mechanical energy between the initial state and the final state where the rod becomes vertical, we obtain the following:

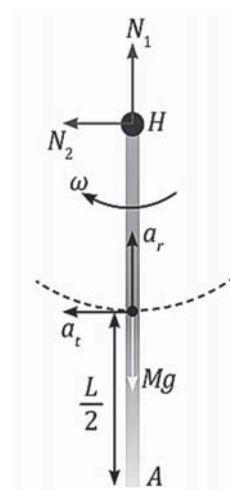
$$K_1 + U_1 = K_2 + U_2$$

$\Rightarrow$ $$0 + 0 = \frac{1}{2}(I_H)\omega^2 + (-Mg\frac{L}{2})$$

$\Rightarrow$ $$\frac{1}{2}(\frac{ML^2}{3})\omega^2 = Mg\frac{L}{2}$$

$\Rightarrow$ $$\omega^2 = \frac{3g}{L}$$

$\Rightarrow$ $$\omega = \sqrt{\frac{3g}{L}}$$



Given that the motion occurs within a vertical plane, hinge forces can manifest in any direction within that plane. Because the direction of the resultant force is uncertain, we opt to decompose it into components (N_1 and N_2) aligned with the directions of radial and tangential acceleration. In the tangential direction,

$$\Sigma F_t = ma_t = m(\alpha r_{\text{com}})$$

$\Rightarrow$ $$N_2 = M(0)\frac{L}{2}$$

$\Rightarrow$ $$N_2 = 0$$

In the radial direction,

$$\Sigma F_r = mg = m(\omega^2 r_{\text{com}})$$

$\Rightarrow$ $$N_1 - Mg = M(\frac{3g}{L})(\frac{L}{2})$$

[$\because$ $\omega = \sqrt{\frac{3g}{L}}$]

$\Rightarrow$ $$N_1 - Mg = \frac{3Mg}{2}$$

$$N_1 = \frac{5Mg}{2}$$

The total hinge force is determined in the following manner:

$$N = \sqrt{N_1^2 + N_2^2} = \frac{5Mg}{2}$$

[$\because$ $N_2 = 0$]

Equations for Pure Rotational Motion:

A system achieves mechanical equilibrium when it maintains both translational and rotational stability.

Translational equilibrium occurs when a body experiences zero net force acting upon it, meaning:

$$\sum \vec{F}_{net} = \vec{0}$$

Rotational equilibrium is achieved when a body experiences zero net torque about any axis, denoted by $\sum \vec{t}_{net} = \vec{0}$.