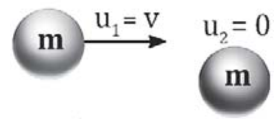
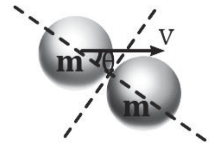
VARIABLE MASS SYSTEM

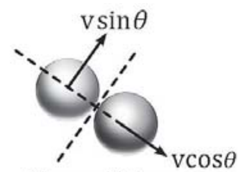
Ex. A ball of mass m makes an elastic collision with another identical ball at rest. Show that if the collision is oblique, the bodies go at the right angles to each other after the collision.



Before collision



During collision



After collision

Sol. Let the initial velocity of the moving ball be v as shown in the figure. We know that the line of impact always passes through the centre when the colliding objects are circular or spherical.

In an elastic unrestricted head-on collision, the particles of the same mass exchange their velocities along the normal at the point of collision. Under identical conditions, in oblique collision, the masses exchange their velocity components along the line of impact.

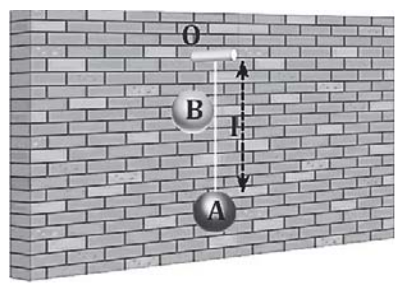
In this problem, the velocity component of the moving ball in the direction of the line of impact is $v\cos\theta$, whereas the velocity component of the stationary ball is zero. So, these two components will be exchanged. Thus, the initial moving ball will retain its velocity component along the tangential direction, $v\sin\theta$, which will be perpendicular to the final velocity of the other ball.

In a completely inelastic collision,

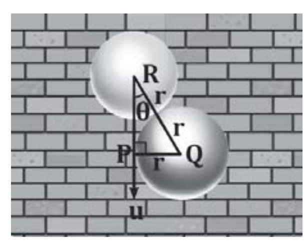
1. The velocity of separation of the colliding bodies along the line of impact is zero.

2. The highest amount of kinetic energy is dissipated in the inelastic collision.

Ex. A small ball A is suspended by an inextensible thread of length l = 1.5 m from O. Another identical ball is thrown vertically downwards such that its surface remains just in contact with the thread during the downward motion and collides elastically with the suspended ball. If the suspended ball just completes a vertical circle after the collision, calculate the velocity of the falling ball just before the collision. ($g = 10\ ms^{-2}$)



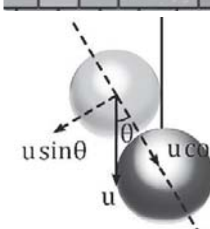
Sol. The velocity vector of ball B is tangential to ball A and both the balls are identical. At the instant of collision, the angle θ can be found from triangle PQR.



We have the following:
$$\sin\theta = \frac{r}{r+r} = \frac{1}{2}$$
$$\theta = 30°$$

If it was an unrestricted collision, the velocity components would have got exchanged along the line of impact. However, the ball A is tied by a string and hence, it will behave as a pendulum. Thus, it can move only in a circular motion.

For the motion of ball A to be circular, the velocity of ball A, at the lowest point, just after the collision, should be horizontal as shown in the figure.

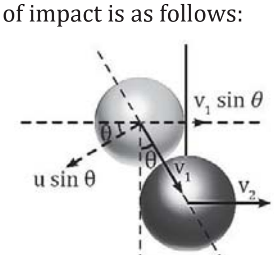


Before collision

The suspended ball A just completes a vertical circle motion after the collision, it should have the critical velocity v_2 (along the horizontal direction) at the lowest position. Critical velocity, $v_2 = \sqrt{5gl}$

As the collision is elastic, the coefficient of restitution along the line of impact is as follows:

$$e = \frac{v_{sep}}{v_{app}}\Big|_{LOI} = 1$$
$$\frac{(v_2\sin\theta - v_1)}{u\cos\theta} = 1$$
$$v_2\sin\theta - v_1 = u\cos\theta$$
Since $\theta = 30°$,
$$\frac{v_2}{2} - v_1 = \frac{\sqrt{3}u}{2}$$
$$\sqrt{3}u + 2v_1 = v_2 \quad \ldots (1)$$



After collision

By using the principle of conservation of momentum in horizontal direction, we get the following:

$$0 + 0 = m[v_1 \sin\theta - (u\sin\theta)\cos\theta] + mv_2$$

$$\frac{v_1}{2} - \frac{\sqrt{3}u}{4} + v_2 = 0$$

$$\sqrt{3}u - 2v_1 = 4v_2 \quad \dots (2)$$

By adding equation (1) and equation (2) we get the following:

$$2\sqrt{3}u = 5v_2$$

$$u = \frac{5v_2}{2\sqrt{3}}$$

$$v_2 = \sqrt{5gl},$$

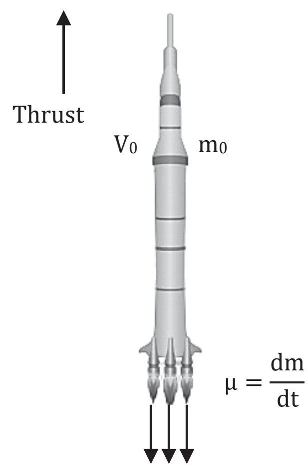
$$u = \frac{5}{2\sqrt{3}}\sqrt{(5)(10)(\frac{3}{2})}$$

$$u = \frac{5}{2\sqrt{3}} \times 5\sqrt{3} = \frac{25}{2}$$

$$u = 12.5\ ms^{-1}$$

Analysis of Rocket Launch

During a rocket launch, thrust is a propulsive force exerted in a particular direction. Let $\mu = \left|\frac{dm}{dt}\right|$ = Constant rate of fuel ejection (kgs^{-1}) as the rocket expends fuel over time, the rate of fuel expulsion becomes negative. $\mu = -\frac{dm}{dt}$

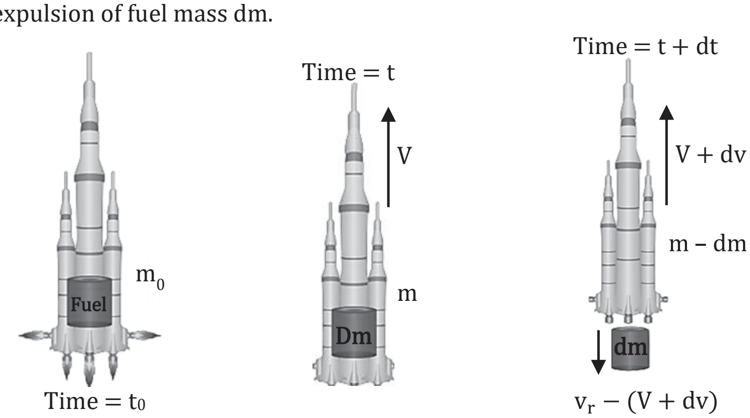


Consider m_0 as the initial total mass of the rocket and its fuel.

If $m =$ instantaneous mass at time t, then $m = m_0 - \mu t$ Let v_o denote the launch velocity of the rocket, and v_r represent the velocity of the ejected fuel relative to the rocket. If the relative velocity of the fuel ejection is extremely low, the rocket will fail to launch. The thrust serves not only to attach or detach mass but also to do so with a certain relative velocity.

At time $t + dt$, suppose the rocket advances along its path with a speed of $v + dv$, and a mass dm of fuel is expelled. The velocity of the expelled fuel relative to the ground is given by $v_r - (v + dv)$.

Because gravity is a continuous force, the system's linear momentum can be conserved both before and after the expulsion of fuel mass dm.



Thus

$$\vec{p}_t = \vec{p}_{t+dt}$$

$$mv = (m - dm)(v + dv) - dm[v_r - (v + dv)]$$

$$mv = mv + m(dv) - (dm)v - (dm)(dv) - (dm)v_r + (dm)v + (dm)(dv)$$

$$0 = mdv - dmv_r$$

$$m\frac{dv}{dt} = (\frac{dm}{dt})v_\gamma$$

As $m\frac{dv}{dt}$ is thrust force, magnitude of this can be written as:

$$F_{Thrust} = v_r\left|\frac{dm}{dt}\right| = \mu v_r$$

As the thrust force balances weight and provides acceleration a to the rocket at time t,

$$\mu v_r - mg = ma = m\frac{dv}{dt}$$

$$\Rightarrow \frac{\mu v_r}{m} - g = \frac{dv}{dt}$$

$$\Rightarrow dv = \left(\frac{\mu v_r}{m} - g\right) dt$$

$$\text{As } m = m_0 - \mu t,$$

$$dv = \frac{\mu v_r}{m_0 - \mu t} dt - g dt$$

Integrating from initial point to time t,

$$\int_{v_0}^{v} dv = \int_0^t \frac{\mu v_r}{m_0 - \mu t} dt - g\int_0^t dt$$

$$[v]_{v_0}^{v} = \mu v_r\left[\frac{1}{-\mu}\ln(m_0 - \mu t)\right]_0^t - g[t]_0^t$$

$$(v - v_0) = -v_r \ln\left|\frac{m_0 - \mu t}{m_0}\right| - g(t - 0)$$

$$v = v_0 + v_r \ln\left|\frac{m_0}{m}\right| - gt$$

The rocket launch is influenced by three components derived from the rocket's velocity equation.

Initial velocity of launch → v_0

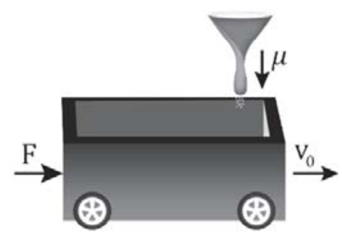
Acceleration due To gravity → gt

Velocity due To thrust → $v_r \ln\left(\frac{m_0}{m}\right)$

$$v = v_0 - gt + v_r \ln\left(\frac{m_0}{m}\right)$$

Flowchart for Thrust Force		
Step 1	Define the system and illustrate the free body diagram (FBD) incorporating the thrust force.	Note Here, m is instantaneous mass
Step 2	Select an axis and decompose the forces along that axis.	
Step 3	Utilize Newton's second law, $F = \sum m\vec{a}$ along every axis.	

Ex. Sand falls from a stationary hopper onto a freight car, which is moving with a uniform velocity of v_0. The sand is falling at a rate of μ. How much force is needed to keep the freight car moving at the speed of v_0?



Sol. Let the sand fall on the car with a velocity v_y.

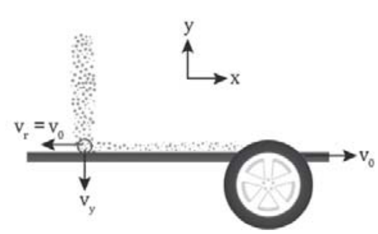
Choose the rightward direction as the positive x-axis.

Given,

Sand falls at the rate of μ

The velocity of sand is as follows:

$$\vec{V}_{sand} = -v_y\hat{j}$$



Velocity of car,

$$\vec{v}_{car} = v_0\hat{i}$$

The relative velocity of sand with respect to car is as follows:

$$\vec{v}_{sand,\,car} = \vec{v}_{sand} - \vec{v}_{car} = -v_0\hat{i} - v_y\hat{j}$$

As there are two components of the relative velocity of sand, the thrust force will also act on car in two directions.

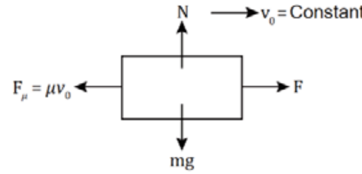
The thrust force acting along x-direction is as follows:

$$F_{T_x} = \mu v_r = \mu v_o$$

The thrust force acting along y-direction is as follows:

$$F_{T_y} = \mu v_y$$

The thrust force acting along y-direction will be balanced by normal reaction and the thrust forcing acting along x-direction will cause the car to accelerate in this direction.



The velocity of sand becomes zero when it falls on the car. for the velocity of car to remain constant, its acceleration along the x-direction should be zero.

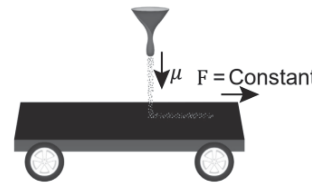
$$\vec{a}_x = 0$$

$$(\sum \vec{F}_{ext})_x = 0$$

Thus, from FBD of car, we get the following:

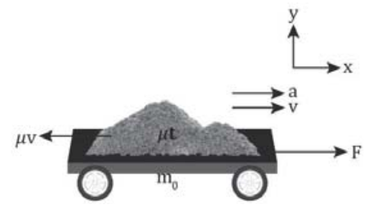
$$F = \mu v_o$$

Ex. A flatcar of mass m_0 starts moving towards the right direction due to a constant horizontal force F. Sand spills on the flatcar from a stationary hopper. The velocity of the loading is constant and equal to μkg s^{-1}. Find the time dependence of the velocity and the acceleration of the flatcar in the process of loading. The friction is negligibly small.



Sol. Let the velocity of the car be v at any time t.

As the sand spills on the car at the rate of μkg s^{-1}, there will be μt kg of sand on the car at time t. From the FBD of the car, we get the following:



$$\sum \vec{F}_{ext} = m\vec{a}$$

$$F - \mu v = (m_0 + \mu t)a$$

$$F - \mu v = (m_0 + \mu t)\frac{dv}{dt}$$

$$\frac{dv}{dt} = \frac{F - \mu v}{m_0 + \mu t}$$

$$\int_0^v \frac{dv}{F - \mu v} = \int_0^t \frac{dt}{m_0 + \mu t}$$

By integrating from time t = 0 to t = t, we get the following:

$$[\frac{1}{-\mu}\ln(F - \mu v)]_0^v = [\frac{1}{\mu}\ln(m_0 + \mu t)]_0^t$$

$$\ln|\frac{F}{F - \mu v}| = \ln|\frac{m_0 + \mu t}{m_0}|$$

$$m_0 F = (F - \mu v)(m_0 + \mu t)$$

$$m_0 F = m_0 F + \mu F t - \mu v m_0 - \mu^2 v t$$

$$0 = Ft - m_0 v - \mu v t$$

$$v(t) = \frac{Ft}{m_0 + \mu t}$$

By differentiating velocity with respect to time using the quotient rule, we get the following:

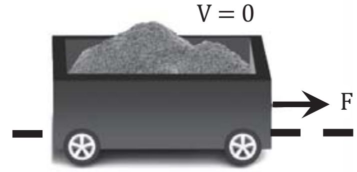
$$\frac{dv}{dt} = a(t) = \frac{(m_0 + \mu t)F - Ft(\mu)}{(m_0 + \mu t)^2}$$

$$a(t) = \frac{m_0 F}{(m_0 + \mu t)^2}$$

Ex. A cart loaded with sand moves on a horizontal road due to a constant force F with its direction coinciding with the cart's velocity vector. In the process, the sand spills through a hole in the bottom with a constant velocity of μkg s^{-1}. Find the velocity of the cart at time t if at the initial moment, the cart loaded with sand had mass m_0 and its velocity was equal to zero. The friction is neglected.

At t = 0, $m = m_0$

V = 0



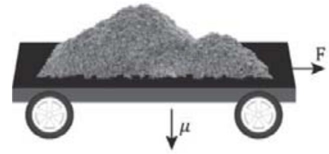
Sol. Let the velocity of the cart be v.

The sand particle that is about to fall will have the same velocity as that of the cart, i.e., the relative velocity of this particle with respect to the cart is zero. Since the relative velocity is zero, the thrust force will also be zero (as the thrust force is, $F_{thrust} = \mu v_r$).

Thus, $$F = ma = (m_0 - \mu t)\frac{dv}{dt}$$

$$\frac{dv}{dt} = \frac{F}{m_0 - \mu t}$$

By integrating from time, t=0 to t=t, we get the following:

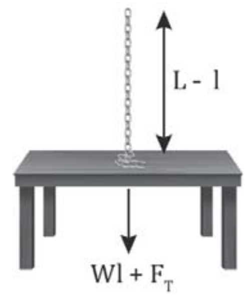


$$\int_0^v dv = F\int_0^t \frac{dt}{m_0-\mu t}$$
$$[v]_0^v = F[\frac{-1}{\mu}\ln(m_0-\mu t)]_0^t$$
$$v = \frac{F}{\mu}\ln[\frac{m_0}{m_0-\mu t}]$$

Ex. A chain of length L and mass M is allowed to fall on a table such that the part falling on the table comes to rest instantaneously. What is the force acting on the table when length l of the chain is on the table?

(A) $\frac{3Mgl}{L}$ (B) $\frac{2Mgl}{L}$ (C) $\frac{Mgl}{L}$ (D) $\frac{2Mgl}{2L}$

Sol. The falling chain applies force on the table. At time t, let the l length of the chain be on the table. At this time, the chain in the air will have the length of (L– l)



Force acting on the table (F) = Weight of the chain lying on the table (Wl) + Thrust force (FT)

Weight of the chain lying on the table (Wl) = $(\frac{M}{L})lg$ ... (1)

The chain is falling under the acceleration due to gravity with the initial velocity of zero.

The velocity of a particle falling from length l is calculated by using the third equation of kinematics.

$$v^2 = u^2 + 2ax$$
$$v_r^2 = 2gl$$

Thrust force is as follows:

$$F_T = \mu v_r = \frac{dm}{dt}v_r$$
$$F_T = (\frac{M}{L})(\frac{dx}{dt})v_r$$
$$F_T = (\frac{M}{L})v_r^2$$
$$F_T = 2(\frac{M}{L})lg$$

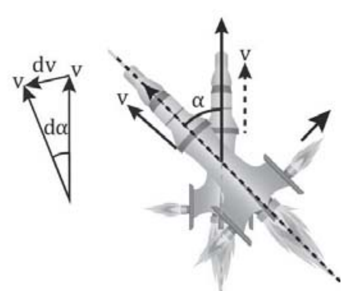
The force acting on the table is as follows:

$$F = W_l + F_T$$
$$F = 2(\frac{M}{L})lg + (\frac{M}{L})lg = \frac{3Mgl}{L}$$

Thus, option (A) is the correct answer.

Ex. A spaceship of mass m_0 moves in the absence of external forces with a constant velocity of v. To change the direction of the motion, a jet engine is switched on. It starts ejecting a gas jet with velocity u, which is constantly relative to the spaceship and directed at an angle of 90° to the motion of the spaceship. The engine is shut down when the mass of the spaceship decreases to m. Through what angle (α) did the direction of motion of the spaceship deviate due to the jet engine operation?

Sol. Here, the engine ejects gas at an angle of 90° to the motion of the spaceship with relative velocity u. Thus, the thrust force is also acting perpendicular, which only changes the direction of the spaceship and not the magnitude of its velocity. The mass of the spaceship changes from m_0 to m in time t. Magnitude of thrust force is as follows:



$|F_{thrust}| = v_r|\frac{dm}{dt}| = u(-\frac{dm}{dt})$[∵ The mass is decreasing]

A small change in velocity corresponding to the change in orientation is as follows:

$$dv = \sqrt{v^2 + v^2 - 2v^2\cos(d\alpha)}$$
$$dv = \sqrt{2v^2[1-\cos(d\alpha)]}$$

$$dv = \sqrt{2v^2 \times 2\sin^2(\frac{d\alpha}{2})} [\because 1 - \cos\theta = 2\sin^2(\frac{\theta}{2})]$$

$$dv = 2v\sin(\frac{d\alpha}{2}) \approx 2v(\frac{d\alpha}{2}) [\because \text{ The angle } d\alpha \text{ is very small.}]$$

$$dv = vd\alpha$$

The thrust force can also be written as follows:

$$F_{thrust} = ma = m\frac{dv}{dt}$$

$$u(\frac{-dm}{dt}) = \frac{mdv}{dt} = \frac{m(vd\alpha)}{dt}$$

By integrating from mass m_0 to m, we get the following:

$$u\int_{m_0}^{m} \frac{-dm}{m} = v\int_0^{\alpha} d\alpha$$

$$u[-\ln(m)]_{m_0}^{m} = v[\alpha]_0^{\alpha}$$

$$u\ln(\frac{m_0}{m}) = v\alpha$$

$$\alpha = \frac{u}{v}\ln(\frac{m_0}{m})$$