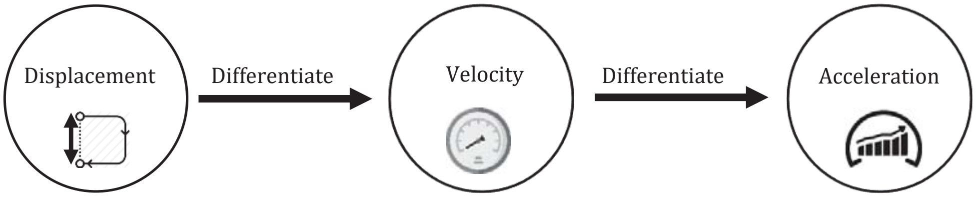
LINEAR MOMENTUM AND CONSERVATION OF LINEAR MOMENTUM

Displacement, Velocity and Acceleration of C.O.M

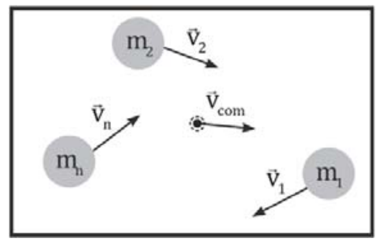


$\Delta\vec{r}_{COM} = \frac{\sum m_i \Delta\vec{r}_i}{M}$ $\vec{v}_{COM} = \frac{\sum m_i \vec{v}_i}{M}$ $\vec{a}_{COM} = \frac{\sum m_i \vec{a}_i}{M}$

$M\vec{a}_{COM} = \sum m_i \vec{a}_i$

Linear Momentum

The linear momentum, $(\vec{p})$ the linear momentum of a particle is defined as the product of its mass and velocity, considering the direction of the velocity.



$$\vec{p} = m\vec{V}$$

The momentum of an n-particle system is the summation of the individual momenta of all n particles, considered as a vector sum.

$$\vec{p}_{sys} = \vec{p}_1 + \vec{p}_2 + \cdots . + \vec{p}_n$$

$$\vec{p}_{sys} = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots ... + m_n\vec{v}_n$$

Also, $\vec{p}_{sys} = M\vec{v}_{cam}$

Where, $M = m_1 + m_2 + \cdots ... + m_n$

Newton's Second Law of Motion

The change in momentum of a system over time equals the net external force acting on the system, and it aligns with the direction of that force. It's established that the acceleration of a system's center of mass is determined by the following expression:

$$\vec{a}_{com} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots ... + m_n\vec{a}_n}{m_1 + m_2 + \cdots .. + m_n} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots ... + m_n\vec{a}_n}{M}$$

$$M\vec{a}_{com} = m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots ... + m_n\vec{a}_n \quad ... (1)$$

Also the net external force on the system is given by,

$$\sum\vec{F}_{sys} = \vec{F}_1 + \vec{F}_2 + \cdots .. + \vec{F}_n = m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots .. + m_n\vec{a}_n. \quad ... (2)$$

Equating equations (1) and (2), we get,

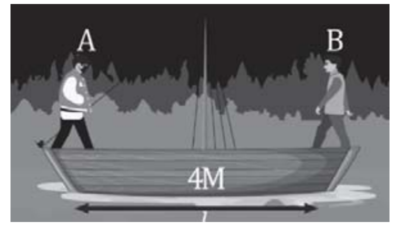
$$\sum\vec{F}_{sys} = M\vec{a}_{com}$$

By Newton's second law of motion, we get the following:

$$\sum\vec{F}_{sys} = M\vec{a}_{com} = \frac{d\vec{p}_{sys}}{dt}$$

Note: When the net external force acting on a system is zero, the velocity of its center of mass remains either zero or constant.

Ex. In a boat of mass 4M and length l on a frictionless water surface, two men A and B of masses M and 2M, respectively, are standing at two opposite ends. Now, A travels a distance of $\frac{1}{4}$ that is relative to the boat towards its centre, and B moves a distance of $\frac{3l}{4}$ and meets A. Find the distance travelled by the boat on the water when they meet.



Sol. Given,

Mass of the boat, $m_3 = 4M$

Length of the boat = l

Mass of man A, $m_1 = M$

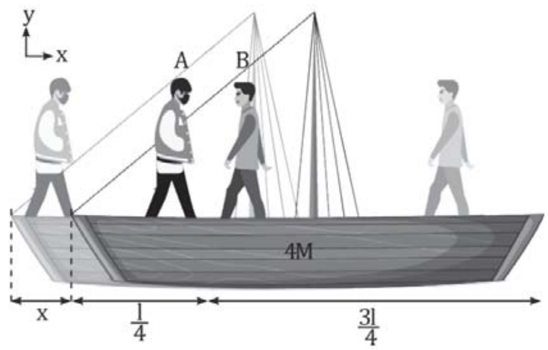
Mass of man B, $m_2 = 2M$

Distance travelled by A, $x_1 = \frac{l}{4}$

Distance travelled by B, $x_2 = \frac{3l}{4}$

When A is walking rightwards, he pushes the boat leftwards, and when B is walking leftwards, he pushes the boat rightwards. Since B has walked more distance, the net displacement of the boat is in the right direction.

Let the distance travelled by the boat when the two men meet be x.



Consider the boat and the two persons together as a system. As there is no net external force acting on the system along the x-direction,

$$(\vec{F}_x)_{net} = \vec{0}$$

$$(\vec{a}_x)_{com} = \vec{0}$$

Since the system was initially at rest, we have,

$$\Delta\vec{x}_{com} = 0$$

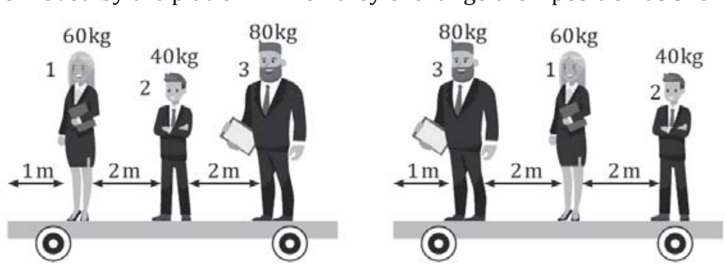
$$\frac{m_1\Delta\vec{x}_1 + m_2\Delta\vec{x}_2 + m_3\Delta\vec{x}_3}{m_1 + m_2 + m_3} = 0$$

$$\frac{M(\frac{l}{4}+x) + 2M(\frac{-3l}{4}+x) + 4M(x)}{M+2M+4M} = 0$$

$$\frac{7MX - \frac{5Ml}{4}}{7M} = 0$$

$$x = \frac{5l}{28}$$

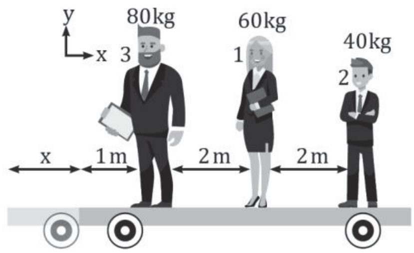
Ex. Three people are standing on a horizontal platform of mass 170 kg as shown in the figure. Find the distance moved by the platform when they exchange their position as shown in the figure.



Sol. Given,

Mass of the platform, $m_4 = 170$ kg

Let the distance travelled by the platform be x when they interchange their positions. Consider the platform and the three persons together as a system.



As there is no net external force acting on the system along the x-direction,

$$\left(\vec{F}_x\right)_{net} = \vec{0}$$

$$\Rightarrow \left(\vec{a}_x\right)_{com} = \vec{0}$$

Since the system was intially at rest we have,

$$\Delta\vec{x}_{com} = 0$$

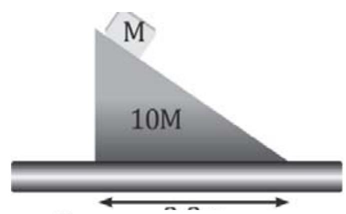
$$\frac{m_1\Delta\vec{x}_1+m_2\Delta\vec{x}_2+m_3\Delta\vec{x}_3+m_4\vec{x}_4}{m_1+m_2+m_3+m_4} = 0$$

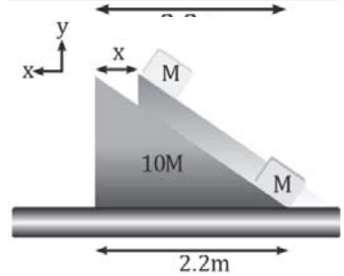
$$\frac{60(2+x)+40(2+x)+80(-4+x)+170x}{60+40+80+170} = 0$$

$$\frac{120+80-320+350x}{350} = 0$$

$$x = \frac{12}{35}\,m$$

Ex. A block of mass M is kept on the top of a larger block of mass 10M. All the surfaces are frictionless. As the system is released from rest, find the distance moved by the larger block at the instant when the smaller block reaches the ground.





Sol. Given,

Mass of the smaller block, $m_1 = M$

Mass of the larger block, $m_2 = 10M$

As the smaller block moves downwards, the larger block will move leftwards. Let the larger block move x distance in the left direction. Consider the two blocks together as a system.

As there is no net external force acting on the system along the x-direction,

$$(\vec{F}_x)_{net} = \vec{0}$$

$$(\vec{a}_x)_{com} = \vec{0}$$

Since the system was initially at rest, we have,

$$\Delta\vec{x}_{com} = 0$$

$$\frac{m_1\Delta\vec{x}_1+m_2\Delta\vec{x}_2}{m_1+m_2} = 0$$

$$\frac{M(-2.2+x)+10M(x)}{M+10M} = 0$$

$$x = \frac{2.2}{11}\,m = 0.2m$$

Ex. A bead can slide on a smooth straight wire, and a particle of mass m is attached to the bead by a light string of length l. The particle is held in contact with the wire with the string being taut as shown in the figure, and then it lets the particle fall. If the bead has a mass of 2m, find the distance it slides when the string makes an angle θ with the wire.

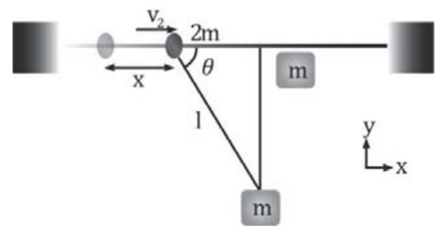
Sol. Given,

Length of the string = l

Mass of the bead = 2m

Mass of the particle = m

Let the bead slide a distance x when the string makes an angle θ with the wire. Consider the bead and particle together as a system.



As there is no net external force acting on the system along the x-direction,

$$(\vec{F}_x)_{net} = \vec{0}$$
$$(\vec{a}_x)_{com} = \vec{0}$$

Since the system was initially at rest, we have,

$$\Delta\vec{x}_{com} = 0$$
$$\frac{m_1\Delta\vec{x}_1 + m_2\Delta\vec{x}_2}{m_1+m_2} = 0$$
$$\frac{m(-(I - I\cos\theta)+x)+2mx}{m+2m} = 0$$
$$I(\cos\theta - 1) + 3x = 0$$
$$x = \frac{I}{3}(1-\cos\theta)$$

Conservation of Linear Momentum

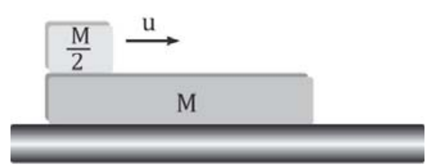
If there is no net external force acting on a system in a specific direction, the system's linear momentum remains constant in that direction.

If $\vec{F}_{ext} = \vec{0} \Rightarrow (\vec{p}_i)_{sys} = (\vec{p}_f)_{sys}$

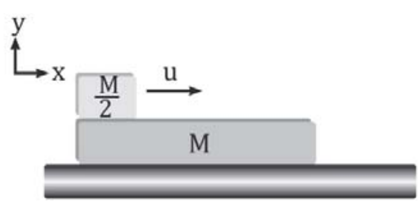
Along the axes,

$$(\vec{F}_x)_{ext} = \vec{0} \Rightarrow (\vec{p}_i)_x = (\vec{p}_f)_x$$
$$(\vec{F}_y)_{ext} = \vec{0} \Rightarrow (\vec{p}_i)_y = (\vec{p}_f)_y$$
$$(\vec{F}_z)_{ext} = \vec{0} \Rightarrow (\vec{p}_i)_z = (\vec{p}_f)_z$$

Ex. A long block of mass M is at rest on a smooth horizontal surface. A small block of mass $\frac{M}{2}$ is placed on the long block and projected with some velocity u. The coefficient of friction between the blocks is μ. Find the velocity of the centre of mass of the system when the system reaches the common velocity.



Sol. As kinetic friction is acting between the blocks, the small box slows down and the long block speeds up until both the blocks reach a common velocity. After this, static (self-adjusting) friction starts acting between them.



Thus, after achieving the common velocity, both the blocks move together with that velocity. Let the common velocity attained by the system be v.

As there is no external force acting on the system along the x-direction,

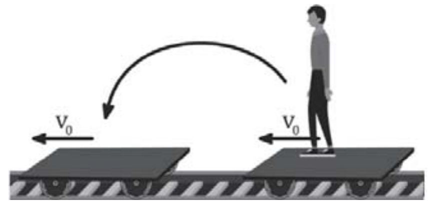
If $(\vec{F}_x)_{ext} = \vec{0} \Rightarrow (\vec{p}_i)_x = (\vec{p}_f)_x$

$$\left(\frac{M}{2} \times u\right) + 0 = \left(\frac{M}{2} + M\right)v$$
$$v = \frac{u}{3}$$

The velocity of the centre of mass at this instant is given by,

$$v_{com} = \frac{m_1v_1 + m_2v_2}{m_1+m_2} = \frac{\frac{M}{2}v + Mv}{\frac{M}{2}+M} = v = \frac{u}{3}$$

Ex. Two identical buggies move one after the other with the same velocity v_0. A man of mass m rides the rear buggy. At a certain moment, the man jumps into the front buggy with velocity u relative to his buggy. Knowing that the mass of each buggy is equal to M, find the velocities with which the buggies will move after the man jumps.



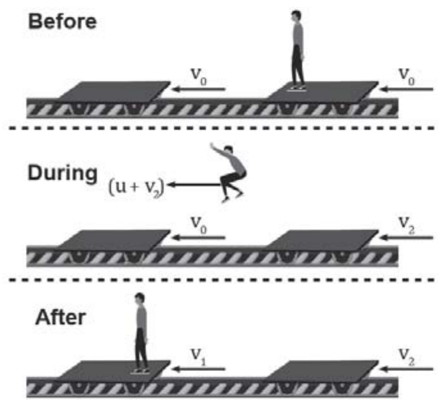
Sol. Given, Initial speed of the two buggies = v_0

Mass of each buggy = M

Mass of the man = m

Let, when the man jumps from one buggy to another, the velocity of the first buggy decrease to v_2 as he pushes it backwards.

Upon his landing on the second buggy, let the velocity of that buggy become v_1.



Consider the man and the first buggy as a system during the jump.

The net external force acting on the system along the x-direction is zero and thus, the momentum is conserved along the x-direction.

Using principle of conservation of momentum for this system before and during the jumping instance, we get the following:

$$(\vec{F}_x)_{ext} = \vec{0} \Rightarrow (\vec{p}_i)_x = (\vec{p}_f)_x$$

$$(M + m)v_0 = m(u + v_2) + MV_2$$

$$v_2 = v_0 - \frac{mu}{M+m} \quad \text{... (1)}$$

Now, consider the man and second buggy as a system.

Conserving momentum along the x-direction during and after the jumping instance.

$$(\vec{F}_x)_{ext} = \vec{0} \Rightarrow (\vec{p}_i)_x = (\vec{p}_f)_x$$

$$(u + V_2)m + MV_0 = (M + m)v_1$$

Substituting value of v_2 from equation (1),

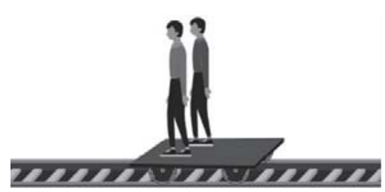
$$(u + v_0 - \frac{mu}{M+m})m + Mv_0 = (M + m)v_1$$

$$Mv_0 + mu + mv_0 - \frac{m^2u}{M+m} = (M + m)v_1$$

$$(M + m)v_0 + mu[1 - \frac{m}{M+m}] = (M + m)v_1$$

$$v_1 = v_0 + \frac{Mm}{(M+m)^2}u$$

Ex. Two men, each of mass m, stand on the edge of a stationary buggy of mass M. Assuming the friction to be negligible, find the velocity of the buggy after both the men jump off the buggy with the same horizontal velocity u relative to the buggy in the following conditions:



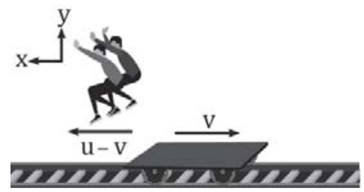
(a) Simultaneously **(b)** One after the other

Sol. Given,

Mass of the each man = m

Mass of the buggy = M

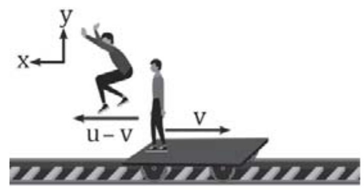
Jumping velocity of the men with respect to the buggy = u



(a) When they jump simultaneously

$$(\vec{F}_x)_{ext} = \vec{0} \Rightarrow (\vec{p}_i)_x = (\vec{p}_f)_x$$
$$(2m + M) \times 0 = -MV + 2m(u - V)$$
$$2mu = 2mV + MV$$
$$V = \frac{2mu}{2m+M}$$

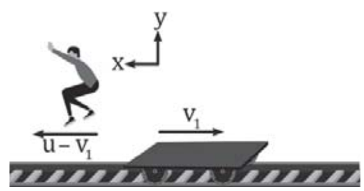
(b) When they jump one after the other Let v be the velocity of the buggy after the first man jumps. Consider the two men and the buggy as a system. As there is no net external force on the system along the horizontal direction, the momentum is conserved. After the first man jumps,



$$(\vec{F}_x)_{ext} = \vec{0} \Rightarrow (\vec{p}_i)_x = (\vec{p}_f)_x$$
$$(2m + M) \times 0 = m(u - V) - (M + m)V$$
$$mu = 2mV + MV$$
$$V = \frac{mu}{2m+M}$$

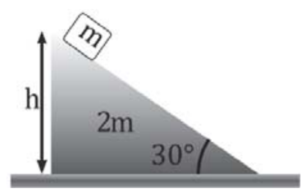
After the second man jumps,

Let v_1 be the velocity of the buggy after the second man jumps.



$$(\vec{F}_x)_{ext} = \vec{0} \Rightarrow (\vec{p}_i)_x = (\vec{p}_f)_x$$
$$-(m + M)v = m(u - v_1) - (M)v_1$$
$$mu + (m + M)v = mv_1 + Mv_1$$
$$v_1 = \frac{mu}{m+M} + v = \frac{mu}{m+M} + \frac{mu}{2m+M}$$
$$(\text{ Since, } v = \frac{mu}{2m+M})$$
$$v_1 = \frac{mu(2M+3m)}{(M+m)(M+2m)}$$

Ex. A block of mass m is placed on a triangular block of mass 2m, which in turn is placed on a horizontal surface as shown in figure. Assuming a frictionless surface, find the velocity of the triangular block when the smaller block reaches the bottom end.



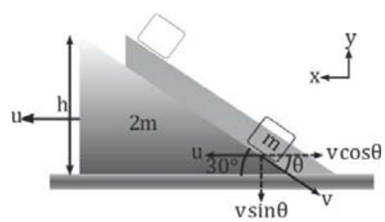
Sol. Given,

Mass of the smaller block = m

Mass of the larger triangular block = 2m

Initial height of the smaller block = h

Let v be the final velocity of the small block with respect to the larger block. Let u be the velocity of the larger block when the small block reaches the bottom position.



As the net external force along the x-direction is zero, the momentum is conserved along the x-direction.

$$(\vec{F}_x)_{ext} = \vec{0} \Rightarrow (\vec{p}_i)_x = (\vec{p}_f)_x$$
$$0 + 0 = 2mu + m(u - v\cos\theta)$$
$$3mu = mv\cos\theta$$
$$3u = v\cos\theta \quad \dots (1)$$

Also, the resultant velocity of the smaller block of mass m is as follows:

$$v_{small} = \sqrt{u^2 + v^2 + 2uv \times \cos(\pi - \theta)}$$

Conserving the mechanical energy of the system (both the blocks together),

$$K_i + U_i = K_f + U_f$$

$$0 + mgh = (\frac{1}{2} \times 2m \times u^2) + \frac{1}{2} \times m \times (u^2 + v^2 + 2uv\cos(\pi - \theta)) + 0$$

$$2gh = 2u^2 + u^2 + v^2 - 2uv\cos\theta$$

Putting value of v from equation (1), we get,

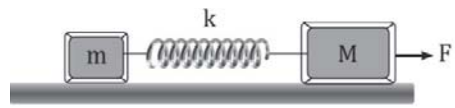
$$2gh = 3u^2 + (\frac{3u}{\cos\theta})^2 - 2u \times \frac{3u}{\cos\theta} \times \cos\theta$$

$$2gh = 3u^2 + (\frac{3u}{\cos 30°})^2 - 6u^2$$

$$2gh = -3u^2 + 12u^2$$

$$u = \frac{\sqrt{2gh}}{3}$$

Ex. A block of mass m is connected to another block of mass M through a spring of negligible mass and spring constant k. The blocks are kept on a smooth horizontal plane and are at rest. The spring is upstretched when a constant force F starts acting on the block of mass M to pull it. Find the maximum extension of the spring.



Sol. Consider both the blocks together as a system.

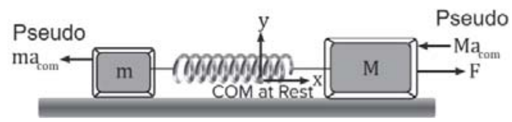
Let a be the acceleration of the centre of mass of the system.

By Newton's second law of motion,

$$a_{com} = \frac{F}{M+m}$$

Centre of mass frame

Due to the acceleration of the centre of mass in the ground frame, the pseudo force comes into the picture that acts as shown in the figure



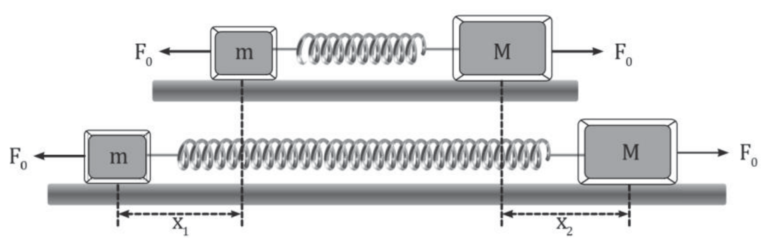
The net rightward force on the M block is,

$$F - Ma_{com} = F - M \times \frac{F}{m+M} = \frac{mF}{m+M}$$

The net leftward force on the m block is,

$$ma_{com} = \frac{mF}{m+M}$$

Therefore, the net horizontal force on the system is zero.



Let, $F_0 = \frac{mF}{m+M}$

Let the extension in both the blocks be x_1 and x_2 respectively as shown in the figure.

Applying work energy theorem, we get,

$$W_{spring} + W_{ext} + W_g + W_N = \Delta KE$$

$$\frac{-1}{2}k[(x_1 + x_2)^2 - 0^2] + F_0(x_1 + x_2) = 0$$

$$F_0(x_1 + x_2) = \frac{k}{2}(x_1 + x_2)^2$$

$$x_1 + x_2 = \frac{2F_0}{k} = \frac{2mF}{k(m+M)}$$

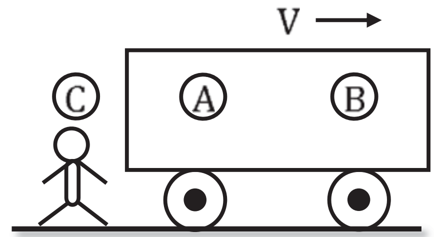
$$x_{max} = \frac{2mF}{k(m+M)}$$

C.O.M Reference Frame

In physics, a frame of reference comprises an abstract coordinate system and a collection of physical reference points. These elements serve to uniquely establish the coordinate system and standardize measurements within that frame.

Let us examine the depicted figure:

If we inquire about B's velocity to A, he would indicate that B is stationary. However, if the same question is posed to C, they would describe B as moving with a velocity V in the positive X direction. Thus, it's evident that before determining velocity, we must establish the frame of reference.



Types of Frame of Reference

Once we have selected our frame of reference, it can fall into one of two categories:

1. **Inertial Frame of Reference**
2. **Non-inertial Frame of Reference**

1. Inertial Frame of Reference

An inertial frame of reference is characterized by the validity of Newton's laws. In such a frame, if no external force acts on a body, it will either remain at rest or continue in uniform motion. For instance, consider a body placed on Earth's surface: from the perspective of an observer on Earth, the body is stationary, whereas from the viewpoint of someone on the Moon, it is in motion. In this scenario, which frame is inertial?

The designation of an inertial frame is actually relative. Initially, we designate a reference frame as the inertial frame of reference. Thus, a more general definition of an inertial frame would be: An inertial frame is one that is either at rest or moving with constant velocity relative to our assumed inertial reference frame.

2. Non-inertial Frame of Reference

Presently, we may delineate a non-inertial frame as a frame subject to acceleration concerning the assumed inertial frame of reference. Within these frames, Newton's law loses validity. Referring back to our previous example, if we designate Earth as the inertial reference frame, the Moon emerges as a non-inertial reference frame due to its accelerated movement concerning Earth. However, to ensure the compliance of Newton's law in this context, we must introduce enigmatic forces, also recognized as pseudo-forces.