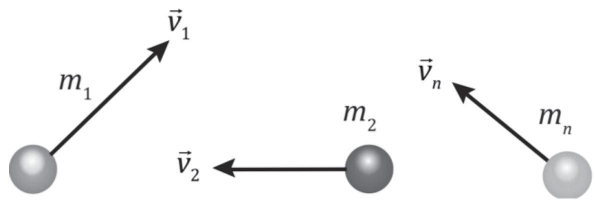
WORK ENERGY THEOREM

Work Done By Internal Forces

Kinetic Energy

Kinetic energy is the energy a system possesses due to its motion. It is a scalar quantity, lacking directionality.

$$KE = \frac{1}{2}m(\vec{v}\cdot\vec{v}) = \frac{1}{2}mv^2$$



SI Unit: joule (J)

If there are n bodies with masses m_1, m_2...m_n moving with velocities $v_1, v_2, ..., v_n$ respectively, then the total kinetic energy of the system can be expressed as follows:

$$KE_{sys} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \cdots \ldots\ldots + \frac{1}{2}m_nv_n^2$$

Note: The work performed by an internal force may be positive, zero, or negative.

Illustrations Of Kinetic Energy And Work Energy Theorem

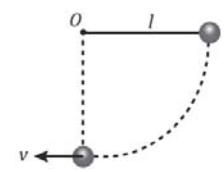
Work Energy Theorem

The theorem asserts that the total work exerted on a system is equivalent to the alteration in its kinetic energy.

$$W_{total} = \int dK = \int \vec{F}\cdot d\vec{r}$$

$$W_{total} = \Delta K = K_2 - K_1 = \frac{1}{2}m(v_f^2 - v_i^2)$$

Ex. When the string of a simple pendulum, with a length of l, is horizontal, the bob is released. Determine its velocity at the lowest point.



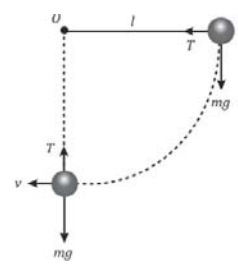
Sol. Let the speed of bob at the bottom be v. Initial speed, u = 0 (since it is released from rest)

There are two forces acting on the bob: Tension and gravity

Let W_T and W_g be the work done by the tension and gravity, respectively, then the total work done by all the forces is given as follows:

$W_{total} = W_T + W_a = 0 + mgl$ (Because tension force is always perpendicular to displacement in this motion)

Applying the work-energy theorem, we get,

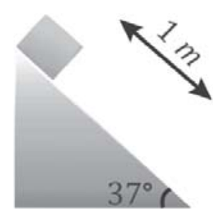


$$W_{total} = \Delta K = \frac{1}{2}mv^2 - 0$$

$$\frac{1}{2}mv^2 = mgl$$

$$v = \sqrt{2gl}$$

Ex. A block of mass 250 g slides down an incline of inclination 37° with a uniform speed. Find the work done against friction as the block slides through 1.0 m. (Take $g = 10\ ms^{-2}$).



Sol. Given,

Mass of the block (m) = 250 g = 0.25 kg

Length through which the block slides (l) = 1.0 m

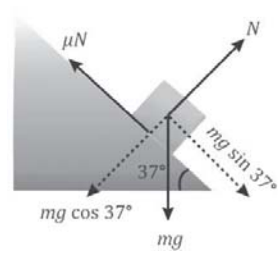
Since the block is moving with a constant velocity, acceleration is zero.

By Newton's second law, if the acceleration is zero, then the net external force acting on the block is also zero.

$$\Sigma\vec{F}_{external} = \vec{0}$$

As the block moves with uniform velocity,

Final velocity = Initial velocity = v



. As the normal reaction is perpendicular to displacement, work done by it is zero. Applying the work-energy theorem in the ground frame, We get, $W_{total} = \Delta K$

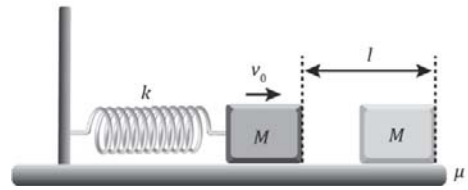
$$W_g + W_N + W_f = \frac{1}{2}m(v^2 - v^2) = 0$$

$$(mg\sin 37^\circ \times 1) + 0 + W_f = 0$$

$$W_f = -mg\sin 37^\circ = -0.25 \times 10 \times \frac{3}{5} = -1.5\ J$$

Work done against friction $= -W_f = 1.5$ J

Ex. A block of mass M attached to a spring (as shown in the figure) initially has a velocity v_0 to the right, and its position is such that the spring exerts no force on it, that is, the spring is not stretched or compressed. The block moves to the right by distance l before stopping as shown in the figure. The spring constant is k and the coefficient of kinetic friction between the block and the table is μ. Find the following when the block moves the distance l:



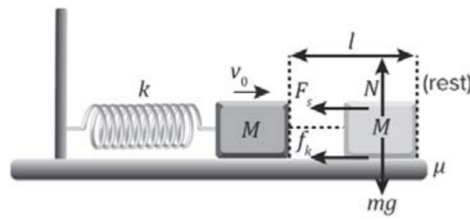
(a) Find the work done on the block by the frictional force.

(b) Find the work done on the block by the spring force.

(c) Are there other forces acting on the block? If yes, what work do they do?

(d) Find the total work done on the block.

(e) Use the work-energy theorem to find the value of l.



Sol. **(a)** Work done by the frictional force balancing the forces in the vertical direction,

We get, $N = Mg$

Also kinetic friction $f_k = \mu N$

Work done by friction is as follows.

$$W_f = \mu Nl\cos 180^\circ = -\mu Mgl$$

(b) Work done by spring force

$$W_{spring} = -\frac{1}{2}k(x_f^2 - x_i^2) = -\frac{1}{2}(l^2 - 0^2)$$

$$W_{spring} = -\frac{1}{2}kl^2$$

(c) Work done by other forces

The other forces acting on the block are gravity and normal reaction force. However, the work done by both gravity and normal forces is zero because they are perpendicular to the displacement during motion.

$$W_g = 0 \text{ and, } W_N = 0$$

(d) Total work done on the block

$$W_{total} = \Delta KE$$

$$W_{total} = \frac{1}{2}M(0^2 - v_0^2)$$

$$W_{total} = -\frac{1}{2}Mv_0^2$$

(e) Finding l

We have, $W_{total} = -\frac{1}{2}Mv_0^2 = W_{spring} + W_f + W_g + W_N$

$$-\frac{1}{2}Mv_0^2 = -\frac{1}{2}kl^2 - \mu Mgl + 0 + 0$$

$$kl^2 + 2\mu Mgl - Mv_0^2 = 0$$

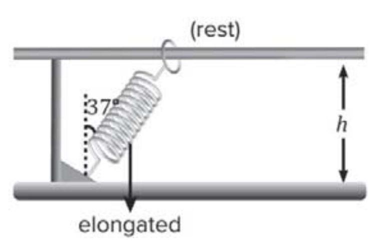
Using a quadratic formula we get,

$$l = \frac{-2\mu Mg \pm \sqrt{(2\mu Mg)^2 - 4 \times k \times (-Mv_0^2)}}{2 \times k}$$

Neglecting negative value of I, we get the following.

$$l = \frac{\mu Mg}{k}\left[\sqrt{1 + \frac{k}{M}\left(\frac{v_0}{\mu g}\right)^2} - 1\right]$$

Ex. One end of a spring of natural length h and spring constant k is fixed at the ground, and the other end is fitted with a smooth ring of mass m that is allowed to slide on a horizontal rod fixed at a height h as shown.



Initially, the spring is elongated and makes an angle of 37° with the vertical when the system is released from rest. Find the speed of the ring when the spring becomes vertical.

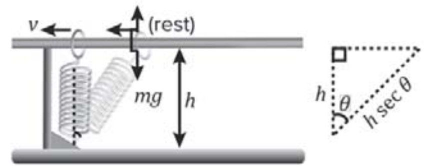
Sol. Given,

Natural length of the spring = h

Spring constant = k

Mass of the ring = m

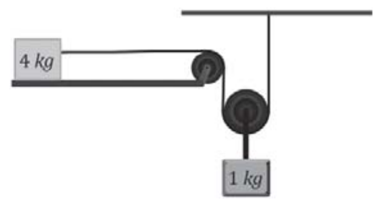
Height of the horizontal rod = h



Let v be the speed of the ring when the spring becomes vertical. Applying the work-energy theorem, we get,

$$W_{total} = W_g + W_N + W_{spring} = \Delta KE$$
$$0 + 0 + W_{spring} = \frac{1}{2}m(v^2 - 0)$$
$$W_{spring} = \frac{1}{2}mv^2$$
$$-\frac{1}{2}k(0^2 - (h\sec\theta - h)^2) = \frac{1}{2}mv^2$$
$$kh^2(\sec\theta - 1)^2 = mv^2$$
$$kh^2(\sec 37° - 1)^2 = mv^2$$
$$kh^2(\frac{1}{16}) = mv^2$$
$$v = \frac{h}{4}\sqrt{\frac{k}{m}}$$

Ex. Consider the situation shown in figure. The system is released from rest and the block of mass 1 kg is found to have a speed 0.3 ms^{-1} after it has descended through a distance of 1 m. Find the coefficient of kinetic friction between the block and the table.



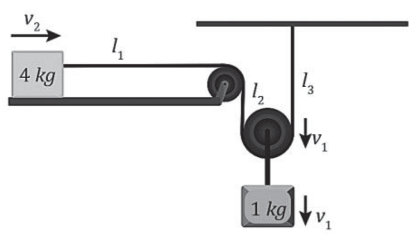
Sol. Let the coefficient of friction between the block and the table be μ.

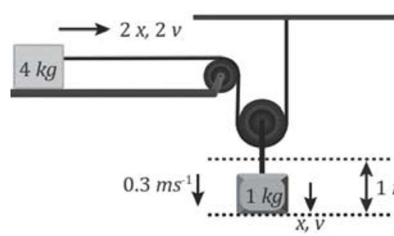
Assuming the intercepts and velocities as shown in the figure.

Applying the constraint equation, we get

$$l_1 + l_2 + l_3 = 0$$
$$-v_2 + v_1 + v_1 = 0$$
$$v_2 = 2v_1 \quad \dots (1)$$

Thus, motion parameters (displacement and velocity) of 4 kg block are double the value of motion parameters of block 1 kg.





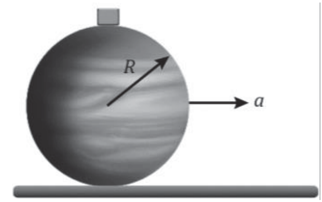
Applying the work-energy theorem on system of two blocks simultaneously, we get,

$$W_{total} = \Delta KE$$
$$W_N + W_g + W_T + W_f = \Delta KE$$

However, $W_N = 0$ because the normal reaction is perpendicular to the displacement. Also, the tension force is an internal force in the chosen system, so $W_T = 0$

$$W_g + W_f = \Delta KE$$
$$[0 + (1 \times gx)] + [(-4 \times \mu g \times 2x) + 0] = [(\frac{1}{2} \times 1 \times v^2) + (\frac{1}{2} \times 4 \times (2v)^2)] - (0 + 0)$$
$$gx - 8\mu gx = \frac{17v^2}{2}$$
$$1 - 8\mu = \frac{17v^2}{2gx} = \frac{17 \times 0.3^2}{2 \times 10 \times 1}$$
$$\mu = 0.115$$

Ex. A smooth sphere of radius R is made to translate in a straight line with a constant acceleration a. A particle kept on the top of the sphere is released from there at zero velocity with respect to the sphere. Find the speed of the particle with respect to the sphere as a function of the angle θ it slides.

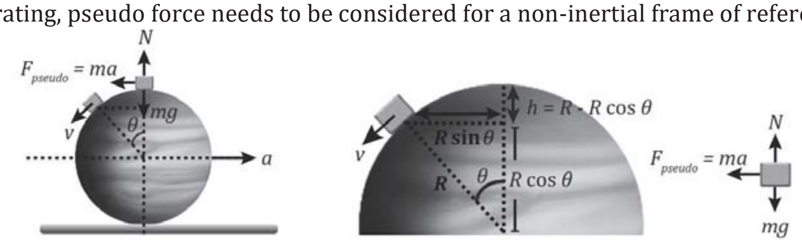


Sol. Given,

Radius of the sphere = R

Acceleration of the sphere = a

Let the speed of the particle with respect to the sphere be v as it slides down. As the sphere is accelerating, pseudo force needs to be considered for a non-inertial frame of reference.



Let, after sliding an angle θ, the particle be at a horizontal distance x and a vertical distance h from the initial point.

$$x = R\sin\theta$$
$$h = R - R\cos\theta$$

Applying the work-energy theorem on the particle in sphere frame, we get,

$$W_g + W_N + W_{pseudo} = \Delta KE$$
$$mgh + 0 + max = \frac{1}{2}m(v^2 - 0^2)$$
$$v = \sqrt{2(gh + ax)} = \sqrt{2[g(R - R\cos\theta)] + aR\sin\theta}$$
$$v = \sqrt{2R[a\sin\theta + g(1 - \cos\theta)]}$$