VERTICAL CIRCULAR MOTION

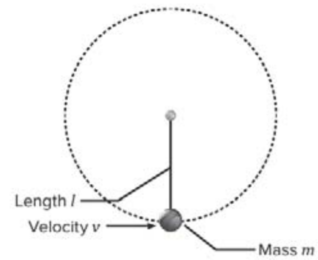
Concept of Vertical Circular Motion

Quadrant Analysis

Vertical circular motion refers to motion in which an object moves in a circular path within a vertical plane.

Let's contemplate a small bob with a mass m attached to an ideal string of length l, restricted to motion within a vertical plane.

At the lowest point, let's impart a velocity v to the bob.



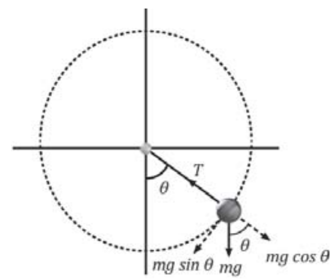
Lower quadrant

At any given time t, let the bob be positioned at an angle θ relative to the vertical, as illustrated in the diagram. Balancing forces in the radial direction,

$$\sum F_c = \frac{mv^2}{R}$$

$$T - mg\cos\theta = \frac{mv^2}{l}$$

$$T = \frac{mv^2}{l} + mg\cos\theta$$



Presently, the weight component consistently endeavors to maintain the string taut in the lower quadrant, ensuring that the tension in the string never reaches zero.

If velocity is zero $T = mg\cos\theta$

Upper quadrant

At any given moment t, let the bob be positioned at an angle θ relative to the vertical, as depicted in the diagram. Balancing forces in the radial direction,

$$\sum F_C = \frac{mv^2}{R}$$

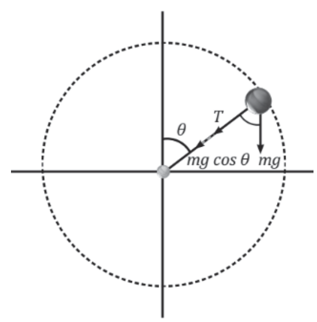
$$T + mg\cos\theta = \frac{mv^2}{l}$$

$$T = \frac{mv^2}{l} - mg\cos\theta$$

$$T = 0$$

$$\frac{mv^2}{l} = mg\cos\theta$$

$$V = \sqrt{gl\cos\theta}$$



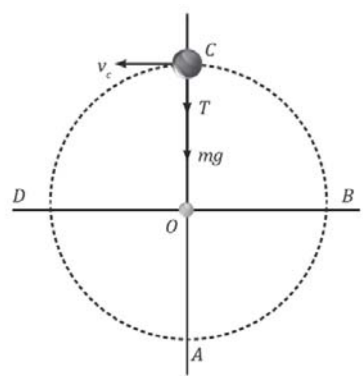
Thus, if the tension becomes zero in the upper quadrant, then the bob will behave like a projectile from that instant.

Critical Circular Motion

Vertical motion that precisely completes a circular path is termed critical circular motion. At the bottom position, let's provide the bob with an initial velocity v.

At the highest point of the circular motion, the component of weight along the center is at its maximum. Therefore, the top position is the critical point in vertical circular motion. This implies that if the bob surpasses the top point, it will undoubtedly complete the circle. Let v_c represent the minimum velocity necessary to surpass the highest point.

Balancing forces in the radial direction at point C,



$$\sum F_C = \frac{mv^2}{R}$$

$$T + mg = \frac{mv_C^2}{l}$$

For criticality, $T \to 0$

$$mg = \frac{mv_C^2}{l}$$

$$v_C = \sqrt{lg}$$

Critical velocity refers to the minimum velocity required at the highest point in vertical circular motion to complete the circle.

Energy Conservation

Let, T_A, T_B, T_C, and T_D be the tension at point A,B,C and D, respectively.

Let, v_A, v_B, v_C, and v_D be the velocity at point A,B,C, and D, respectively.

Let the lowest point A be the datum or the starting point for potential energy.

Thus $$U_A = 0$$

The total energy of the bob is constant for every point in the vertical circle.

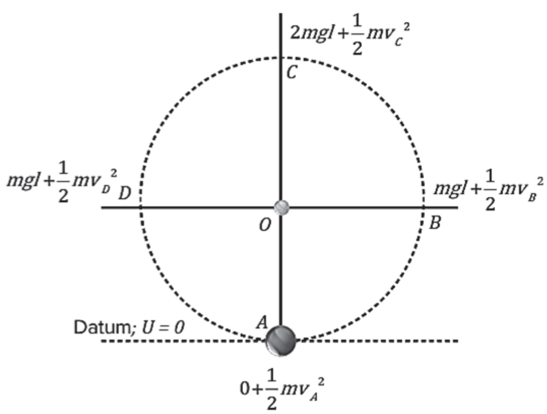
Thus $$E = U + K$$

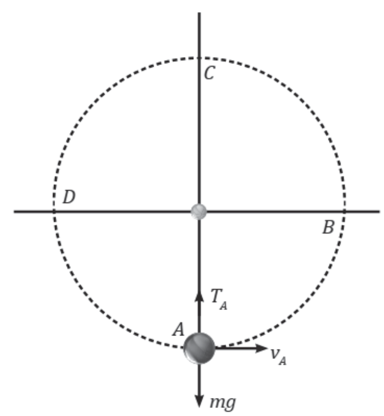
At point A, $$E_A = U_A + K_A$$
$$E_A = 0 + \frac{1}{2}mV_A^2 = \frac{1}{2}mV_A^2$$

At point B, $$E_B = U_B + K_B$$
$$E_B = mgl + \frac{1}{2}mV_B^2$$

At point C, $$E_c = U_c + K_C$$
$$E_C = 2mgl + \frac{1}{2}mv_C^2 = 2mgl + \frac{1}{2}m(\sqrt{gl})^2 = \frac{5mgl}{2}$$

A t point D, $$E_D = U_D + K_D$$
$$E_D = mgl + \frac{1}{2}mV_D^2$$
$$E_A = E_B = E_C = E_D = E$$
$$\frac{1}{2}mV_A^2 = mgl + \frac{1}{2}mV_B^2 = \frac{5mgl}{2} = mgl + \frac{1}{2}mv_D^2$$





On solving, we get, $$V_A = \sqrt{5gl}$$
$$V_B = \sqrt{3gl}$$
$$V_D = \sqrt{3gl}$$

Now, the FBD of the bob at point A is, Balancing forces in the radial direction,
$$\sum F = \frac{mV_A^2}{R}$$
$$T_A - mg = \frac{mV_A^2}{l} = \frac{m(\sqrt{5gl})^2}{l} = 5mg$$
$$T_A = 6mg$$

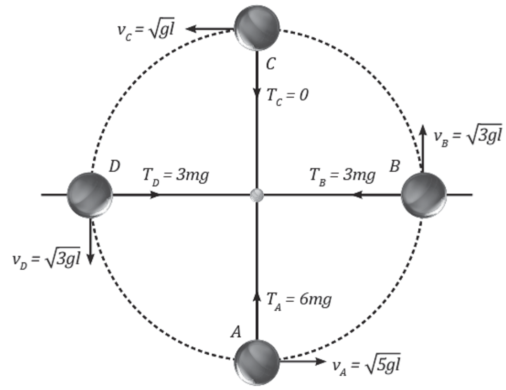
Similarly, for all other points, we have,

At point B, $$T_B = \frac{mV_B^2}{l} = \frac{m(\sqrt{3gl})^2}{l} = 3mg$$

At point C, $$T_C + mg = \frac{mv_C^2}{l} = \frac{m(\sqrt{gl})^2}{l}$$
$$T_C = 0$$

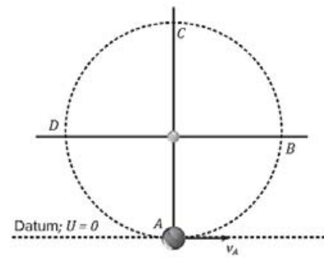
At point D, $$T_D = \frac{mV_D^2}{l} = \frac{m(\sqrt{3gl})^2}{l} = 3mg$$

In case of critical circular motion,



Oscillation

The highest point attainable for the bob during safe oscillation is at the horizontal level. Ensuring the conservation of total energy at points A and B,



We get,

$$U_A + K_A = U_B + K_B$$
$$0 + \frac{1}{2}mV_A^2 = mgl + 0$$
$$V_A = \sqrt{2gl}$$

1. Critical circular motion

Position	Velocity	Tension
A	$\sqrt{5gl}$	6 mg
B	$\sqrt{3gl}$	3 mg
C	$\sqrt{gl}$	0
D	$\sqrt{3gl}$	3 mg

2. Inequalities

Complete Circle	$V_A > \sqrt{5gl}$
Projectile	$\sqrt{2gl} < V_A < \sqrt{5gl}$
Oscillation	$V_A < \sqrt{2gl}$

Ex. In a pendulum, a bob of mass m which was initially at rest is given a sharp hit to impart a horizontal velocity $\sqrt{10gl}$ where l is the length of the pendulum. Find the tension in the string in the following cases:

(a) When the string is horizontal (b) When the bob is at its highest point

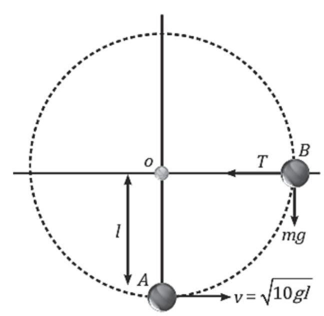
(c) When the string makes an angle of 60° with the upward vertical

Sol. Given,

Mass of the bob = m

Velocity, v = $\sqrt{10gl}$

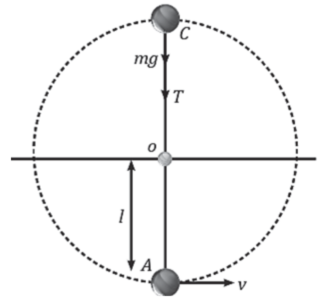
(a) Tension when the string is horizontal Let the lowest initial point be the potential energy datum. Conserving mechanical energy at both the points, we get,



$$K_A + U_A = K_B + U_B$$
$$\frac{1}{2}mV_A^2 + 0 = \frac{1}{2}mV_B^2 + mgl$$
$$V_A^2 = V_B^2 + 2gl$$
$$10gl = V_B^2 + 2gl$$
$$V_B = \sqrt{8gl}$$
$$T_B = \frac{mV_B^2}{l} = 8mg$$

(b) Tension at the highest point

As the initial velocity is more than $\sqrt{5gI}$ the bob will complete the whole circle and the highest point will be C, as shown.



Balancing forces at point C along the radial direction,

$$T_c + mg = \frac{mV_C^2}{l} \quad \text{... (1)}$$

Conserving mechanical energy at points A and C,

$$K_A + U_A = K_C + U_C$$

$$\frac{1}{2}mV_A^2 + 0 = \frac{1}{2}mV_C^2 + 2mg\left(I + \frac{l}{2}\right)$$

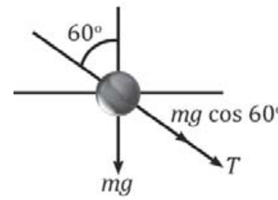
$$V_A^2 = V_C^2 + 3gl$$

$$10gl = V_C^2 + 3gl$$

$$V_C = \sqrt{7gl}$$

$$T_C = \frac{mV_C^2}{l} - \frac{mg}{2} = 6.5mg$$

(c) Tension when the string makes an angle of 60° with the upward vertical

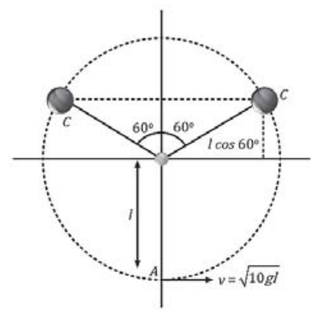


Balancing forces at point C along the radial direction,

$$T_C + mg\cos 60° = \frac{mv_C^2}{l}$$

$$T_C = \frac{mv_C^2}{l} - \frac{mg}{2}$$

Conserving mechanical energy at points A and C, we get,



$$K_A + U_A = K_C + U_C$$

$$\frac{1}{2}mV_A^2 + 0 = \frac{1}{2}mV_C^2 + 2mg\left(l + \frac{l}{2}\right)$$

$$V_A^2 = V_C^2 + 3gl$$

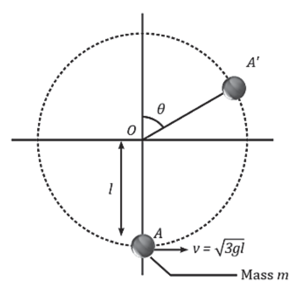
$$10gI = V_C^2 + 3gl$$

$$V_C = \sqrt{7gl}$$

$$T_C = \frac{mV_C^2}{l} - \frac{mg}{2} = 6.5mg$$

Ex. Bob of a stationary pendulum is given a sharp hit to impart a horizontal speed of $\sqrt{3gl}$. Find the angle rotated by the string before it becomes slack.

Sol. Let at point A', which is located at an angle θ from the upward vertical, the string becomes slack. So, the tension of the string at A' is zero. Conserving mechanical energy at points A and A', we get,



$$\frac{1}{2}m(\sqrt{3gI})^2 + 0 = \frac{1}{2}mV_{A'}^2 + mg(I + I\cos\theta)$$

$$3gI = V_{A'}^2 + 2gI(1 + \cos\theta) \quad \text{... (1)}$$

Also, the FBD of the bob is, Balancing forces in the radial direction,

$$T + mg\cos\theta = \frac{mV_{A'}^2}{I}$$

When string is slack, $T = 0$

$$V_{A'}^2 = gl\cos\theta \quad \text{... (2)}$$

Substituting equation (2) in equation (1),

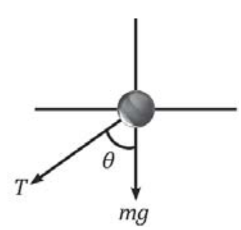
$$3gl = gl\cos\theta + 2gl(1 + \cos\theta)$$

$$3 = \cos\theta + 2 + 2\cos\theta$$

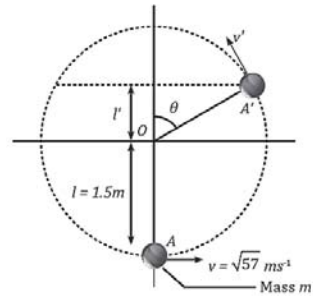
$$\cos\theta = \frac{1}{3}$$

$$\theta = \cos^{-1}\frac{1}{3}$$

Angle rotated $= \pi - \theta = \pi - \cos^{-1}\frac{1}{3}$



Ex. A heavy particle is suspended by a 1.5 m long string. It is given a horizontal velocity of $\sqrt{57}$ ms^{-1}. Find the angle made by the string with the upward vertical when it becomes slack. (Take $g = 10$ ms^{-2}).



Sol. Let at point A', which is located at an angle θ from the upward vertical, the string becomes slack. So, the tension of the string at A' is zero. Conserving mechanical energy at points A and A', we get,

$$\frac{1}{2}m(\sqrt{57})^2 + 0 = \frac{1}{2}mV_{A'}^2 + mg(I + I\cos\theta)$$

$$57 = V_{A'}^2 + 2gl(1 + \cos\theta) \quad \dots (1)$$

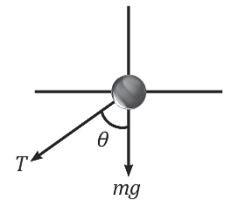
Also, the FBD of the bob is as shown in the second figure,

Balancing forces in the radial direction,

$$T + mg\cos\theta = \frac{mV_{A'}^2}{l}$$

When string is slack, T = 0

$$V_{A'}^2 = gl\cos\theta \quad \dots (2)$$



Substituting equation (2) in equation (1),

$$57 = gl\cos\theta + 2gl(1 + \cos\theta)$$

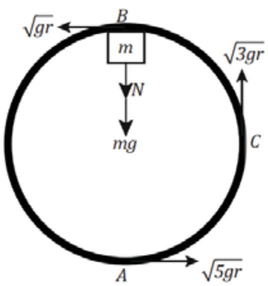
$$57 = gl(3\cos\theta + 2) = 15(3\cos\theta + 2)$$

$$3\cos\theta = \frac{19}{5} - 2 = \frac{9}{5}$$

$$\theta = \cos^{-1}\frac{3}{5} = 53^\circ$$

Applications of Vertical Circular Motion

Case 1: A mass moving on a smooth circular vertical track



Consider a body with mass m moving in a vertical circle, where A denotes the bottom-most point and B denotes the top-most point, as depicted in the figure.

Completing a loop is synonymous with completing a circle.

At point B, along the radial direction,

$$N + mg = \frac{mv^2}{R}$$

In critical case the normal reaction on the body at the highest point will be zero.

Thus
$$0 + mg = \frac{mv^2}{R}$$

$$v = \sqrt{gR}$$

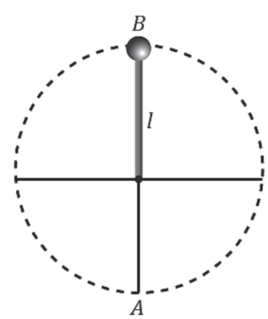
This is the same result as we derived in the vertical circular motion.

Thus, for critical case.

Initial velocity at point = A = $\sqrt{5gR}$

Velocity at the horizontal level = $\sqrt{3gR}$

Case 2: A particle attached to a light rod



Consider a particle of mass m connected to a light rod of length, moving in a vertical circle.

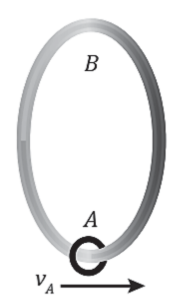
In the critical scenario, the velocity at the highest point is zero, considering the lowest point as the reference for potential energy. By applying the principle of conservation of mechanical energy at both the highest and lowest points, we get:

$$K_A + U_A = K_B + U_B$$

$$\frac{1}{2}mV_A^2 + 0 = \frac{1}{2}m \times 0^2 + mg \times 2l$$

$$v_A = \sqrt{4gl}$$

Case 3: A ring attached to a ring



Consider a small ring connected to a larger ring with a radius of l. The small ring loops around the larger ring to complete vertical circular motion.

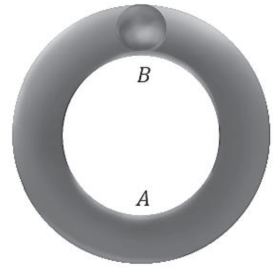
In the critical scenario, the velocity at the highest point is zero. By using the principle of conservation of mechanical energy at both the highest and lowest points, with the lowest point taken as the reference for potential energy, we get:

$$K_A + U_A = K_B + U_B$$

$$\frac{1}{2}mv_A^2 + 0 = \frac{1}{2}m \times 0^2 + mg \times 2l$$

$$v_A = \sqrt{4gl}$$

Case 4: A bead rotated between smooth surfaces

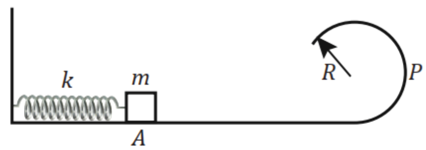


Allow a bead to rotate between the smooth vertical circular surfaces as depicted.

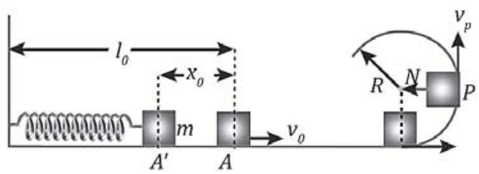
In the critical scenario, the velocity at the highest point is zero. By considering the lowest point as the reference for potential energy and conserving mechanical energy at both the highest and lowest points, we obtain:

$$K_A + U_A = K_B + U_B$$
$$\frac{1}{2}mv_A^2 + 0 = \frac{1}{2}m \times 0^2 + mg \times 2l$$
$$v_A = \sqrt{4gl}$$

Ex. The figure shows a smooth track, a part of which is a circle of radius R. A block of mass m is pushed against a spring of spring constant k, fixed at the left end and is then released. Find the initial compression of the spring so that the block presses the track with a force mg when it reaches the point P, where the radius of the track is horizontal.



Sol. Let the original length of the spring be l_0 and compression required be x_0. Let point A' be the point, where the compression of the spring is x_0.



Balancing the forces along the radial direction at point P, we get,

$$N_p = \frac{mv_p^2}{R}$$

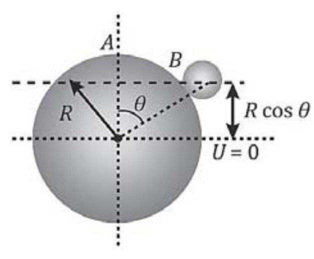
However, $N_p = mg$ (given)

$$mg = \frac{m{v_p}^2}{R}$$
$$v_p = \sqrt{gR} \qquad \text{... (1)}$$

Taking the horizontal track as datum and using the principle of conservation of mechanical energy at points A' and P, we get,

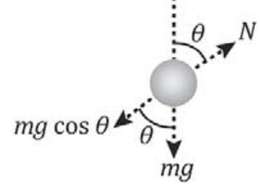
$$K_{A'} + U_{A'} = K_p + U_p$$
$$0 + \frac{1}{2}k{x_0}^2 = \frac{1}{2}m{v_p}^2 + mgR$$
$$\frac{1}{2}k{x_0}^2 = \frac{1}{2} \times m \times (gR) + mgR$$
$$k{x_0}^2 = 3mgR$$
$$x_0 = \sqrt{\frac{3mgR}{k}}$$

Ex. A particle slides on the surface of a fixed smooth sphere starting from the top-most point. Find the angle rotated by the radius through the particle, when it loses contact with the sphere.



Sol. Let the particle slide on a smooth sphere and let it lose contact with the sphere at point B. When it loses contact, then the normal reaction becomes zero, i.e., N = 0. Let the angle rotated by the radius with the vertical when it loses contact with the sphere be θ.

At point B, $$mg\cos\theta - N = \frac{mv^2}{R}$$
$$v^2 = gR\cos\theta \qquad \text{... (1)}$$

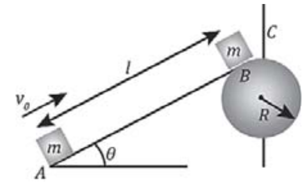


Taking the horizontal line through the centre as the datum for potential energy and using the principle of conservation of mechanical energy at points A and B, we get,

$$K_A + U_A = K_B + U_B$$

$$0 + mgR = \frac{1}{2}mv^2 + mgR\cos\theta$$
$$2gR = Rg\cos\theta + 2gR\cos\theta \text{ (from equation (i))}$$
$$2 = 3\cos\theta$$
$$\theta = \cos^{-1}(\tfrac{2}{3})$$

Ex. The figure shows a smooth track that consists of a straight inclined part of length l joining smoothly with the circular part. A particle of mass m is projected up the incline from the bottom.

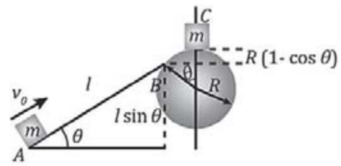


(a) Find the minimum projection speed v0 for which the particle reaches the top of the track.

(b) Assuming that the projection speed is 2v0 and that the block does not lose contact with the track before reaching its top, find the force acting on it when it reaches the top.

Sol. **(a)** Let the minimum speed be v_0 for which the particle reaches the top of the track

Taking the horizontal line through A as the datum and conserving the mechanical energy at points A and C, we get,

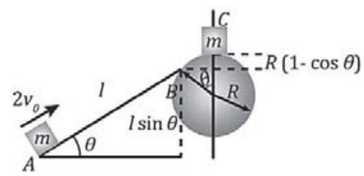


$$K_A + U_A = K_C + U_C$$
$$\frac{1}{2}mv_0^2 + 0 = 0 + mg(l\sin\theta + R(1-\cos\theta))$$
$$v_0 = \sqrt{2g(l\sin\theta + R(1-\cos\theta))}$$

(b) Initial Speed $= 2v_0 = 2\sqrt{2g(l\sin\theta + R(1-\cos\theta))} = \sqrt{8gh}$

$$h = (l\sin\theta + R(1-\cos\theta))$$

Let the force acting on the particle at the top-most point be N. Taking the horizontal line through A as the datum and using the principle of conservation of mechanical energy at points A and C, we get,



$$K_A + U_A = K_C + U_C$$
$$\frac{1}{2}m(2v_0)^2 + 0 = K_C + mgh$$
$$2mv_0^2 - mgh = K_C$$
$$4mgh - mgh = K_C$$
$$K_C = 3mgh = \frac{1}{2}mv^2$$
$$mv^2 = 6mgh \quad \dots (1)$$

Now, from FBD of the particle at the top-most pointC, we get,

$$mg - N = \frac{mv^2}{R}$$
$$N = mg - \frac{6mgh}{R}\left(\text{ from equation (1)}\right)$$
$$N = mg - \frac{6mg}{R}\left(l\sin\theta + R(1-\cos\theta)\right)$$