Potential Energy and Force

Relation between Potential Energy and Force

Let $\overrightarrow{dS}$ represent the displacement experienced by a particle when it is positioned within a region where conservative forces act.

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

$$dU = -W_{cons} = -\vec{F}\cdot d\vec{s}$$

$$dU = -(F_x\hat{i} + F_y\hat{j} + F_z\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$dU = -F_xdx - F_ydy - F_zdz$$

$$dU = -F_xdx - F_ydy - F_zdz$$

$dU = -F_xdx$	$dU = -F_xdx$	$dU = -F_xdx$
If, $dy = 0, dz = 0$	If, $dx = 0, dz = 0$	If, $dx = 0, dy = 0$

Thus, $F_x = -\frac{\partial U}{\partial x}$ $\quad F_y = -\frac{\partial U}{\partial y}$ $\quad F_z = -\frac{\partial U}{\partial z}$

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k}$$

Note: A partial derivative of a function with respect to a variable entails differentiating the function concerning that variable while holding all other variables constant.

Ex. The potential energy of a particle in a space is given by $x^2 + y^2$. Find the force associated with this potential energy.

Sol. Given

Potential energy, $U = x^2 + y^2$

Let $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ be the force associated with this potential energy. We know,

$$F_x = -\frac{\partial U}{\partial x} = -\frac{\partial}{\partial x}(x^2 + y^2) = -2x$$

$$F_y = -\frac{\partial U}{\partial y} = -\frac{\partial}{\partial y}(x^2 + y^2) = -2y$$

$$F_z = -\frac{\partial U}{\partial z} = -\frac{\partial}{\partial z}(x^2 + y^2) = 0$$

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = -2x\hat{i} - 2y\hat{j}$$

Ex. Find the potential energy of the given force, $\vec{F} = -2x\hat{i} - 2y\hat{j}$.

Sol. Given,

Force, $\vec{F} = -2x\hat{i} - 2y\hat{j}$

Let U be the potential energy of the force.

$$U = -\int \vec{F}\cdot d\vec{r} = (-2x\hat{i} - 2y\hat{j})\cdot(dx\hat{i} + dy\hat{j})$$

$$U = \int 2xdx + \int 2ydy = x^2 + y^2 + c$$

Ex. The potential energy function of a particle in a region of space is given as, $U = (2xy + yz)$. Here x, y, and z are in meters. Find the force acting on the particle at a general point P (x, y, z).

Sol. Given,

Potential energy, $U = (2xy + yz)$

Let $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ be the force associated with this potential energy.

$$F_x = -\frac{\partial U}{\partial x} = -\frac{\partial}{\partial x}(2xy + yz) = -2y$$

$$F_y = -\frac{\partial U}{\partial y} = -\frac{\partial}{\partial y}(2xy + yz) = -2x - z$$

$$F_z = -\frac{\partial U}{\partial z} = -\frac{\partial}{\partial z}(2xy + yz) = -y$$

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = -2y\hat{i} + (-2x - z)\hat{j} - y\hat{k}$$

Ex. The potential energy function of a particle in a region of space is given as, $U(x,y) = \sin(x+y)$ Find the force acting on the particle of mass m at point $(0, \frac{\pi}{4})$.

Sol. Given,

Potential energy, $U(x,y) = \sin(x+y)$

Let $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ be the force associated with this potential energy.

$$F_x = -\frac{\partial U}{\partial x} = -\frac{\partial}{\partial x}(\sin(x+y)) = -\cos(x+y) \times (1+0)$$

$$F_y = -\frac{\partial U}{\partial y} = -\frac{\partial}{\partial y}(\sin(x+y)) = -\cos(x+y) \times (0+1)$$

$$F_z = -\frac{\partial U}{\partial z} = -\frac{\partial}{\partial z}(\sin(x+y)) = 0$$

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = -\cos(x+y)\hat{i} - \cos(x+y)\hat{j}$$

At, $(0, \frac{\pi}{4})$,

$$\vec{F} = -\cos(0 + \frac{\pi}{4})(\hat{i} + \hat{j}) = -\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$$

$$|\vec{F}| = \sqrt{(\frac{-1}{\sqrt{2}})^2 + (\frac{-1}{\sqrt{2}})^2} = 1$$

Ex. Find the expression of potential energy U (x, y, z) for a conservative force in a force field given as $\vec{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$ Consider the potential energy at the point (2, 2, 2) as zero.

Sol. Given,

Force, $\vec{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$

Let U be the potential energy for the force.

$$dU = -\vec{F} \cdot d\vec{r} = -(yz\hat{i} + xz\hat{j} + xy\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$dU = -(yzdx + xzdy + xydz) = -d(xyz)$$

$$U = \int dU$$

$$U = -xyz + c$$

However, $U(2,2,2) = 0$

$$0 = -2 \times 2 \times 2 + c$$

$$c = 8$$

$$U = -xyz + 8$$

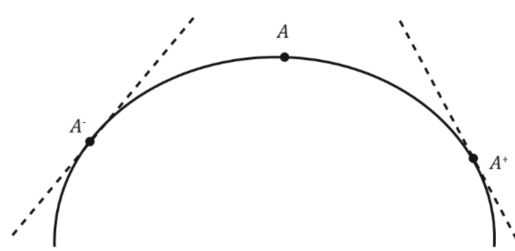
Potential Energy Curve

A plot depicting the potential energy of a particle against its displacement from the center of the force field is referred to as a potential energy curve.

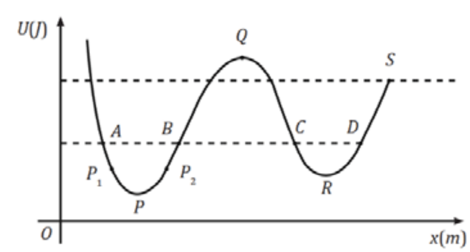
Let $\vec{F} = -\frac{\partial U}{\partial r}\hat{r}$ Be the force acting on a particle

At point A+, the potential energy curve exhibits a negative slope, indicating that the force acts in the positive direction.

At point A-, the potential energy curve has a positive slope, indicating that the force acts in the negative direction.



Let's examine a potential energy curve plotted for a particle moving in the x-direction, as depicted in the provided figure.



Case 1: When the body is positioned at point P, an increase in x leads to an increase in U, indicating a force in the negative x direction. Conversely, a decrease in x results in a decrease in U, signifying a force in the positive x direction. Essentially, the body consistently encounters an attractive force directed towards point P. If displaced either to the left or right from this position, its potential energy increases, compelling the body to return to its state of minimum energy, namely point P. The attractive force aids the body in re-establishing its original position.

Case 2: When the body is positioned at point Q, an increase in x results in a decrease in U, with the force acting in the positive x direction. Conversely, a decrease in x leads to an increase in U, with the force directed in the negative x direction. In essence, the body possesses maximum potential energy at point Q. With further displacement from this point, a repulsive force acts on the body, driving it away from point Q.

Analysis of Equilibrium

Let $\vec{F} = -\frac{dU}{dr}\hat{r}$ be the force acting on a particle.

At equilibrium, the particle experiences a balanced net force, resulting in a state of zero net force acting on it.

$$\vec{F} = 0$$
$$\frac{dU}{dr} = 0$$

Stable equilibrium

When a particle undergoes a slight displacement from its position and a force acting upon it restores it to the original position, it is described as being in a stable equilibrium position.

1. $$\frac{dU}{dx} = 0$$
2. $$\frac{d^2U}{dx^2} = \text{Positive}$$

Unstable equilibrium

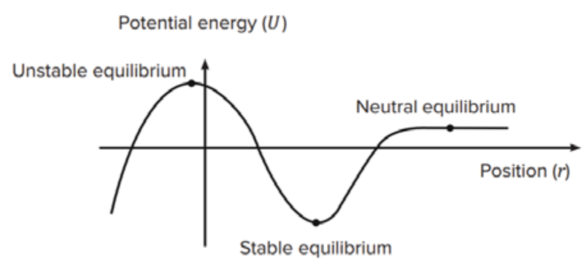
When a particle undergoes a slight displacement from its position and the force acting on it tends to move the particle further away from the equilibrium position, it is referred to as being in an unstable equilibrium.

1. $$\frac{dU}{dx} = 0$$
2. $$\frac{d^2U}{dx^2} = \text{Negative}$$

Neutral equilibrium

In a state of neutral equilibrium, the potential energy remains constant. When a particle is displaced from its position, it does not encounter any force acting upon it, thus maintaining equilibrium in the displaced state. This situation is referred to as neutral equilibrium.

1. $$\frac{dU}{dx} = 0$$
2. $$\frac{d^2U}{dx^2} = 0$$



Ex. The potential energy of a particle varies with x according to the relation, $U(x) = x^2 - 4x$ what is point $x = 2$ a point of?

(A) Stable equilibrium
(B) Unstable equilibrium
(C) Neutral equilibrium
(D) None of the above

Sol. Given,

$$U(x) = x^2 - 4x$$
$$\frac{dU}{dx} = \frac{d}{dx}(x^2 - 4x) = 2x - 4$$
$$\frac{dU}{dx}(2) = 2 \times 2 - 4 = 0$$

Also, $$\frac{d^2U}{dx^2} = \frac{d}{dx}(2x - 4) = 2 = \text{Positive}$$

So, the equilibrium is stable.

Thus, option (A) is the correct answer.

Ex. The potential energy between two atoms in a molecule is given by $U(x)=\frac{a}{x^{12}}-\frac{b}{x^{6}}$ where a and b are positive constants and x is the distance between the atoms. Find the condition when the system is in stable equilibrium.

Sol. Given,

Potential energy, $U(x)=\frac{a}{x^{12}}-\frac{b}{x^{6}}$

For the system to be in stable equilibrium,

$$\frac{dU}{dx}=0,\text{ and }\frac{d^2U}{dx^2}=\text{ Positive}$$

$$\frac{dU}{dx}=\frac{d}{dx}\left(\frac{a}{x^{12}}-\frac{b}{x^{6}}\right)=-12ax^{-13}+6bx^{-7}=0$$

$$bx^{-7}=2ax^{-13}$$

$$x^6=\frac{2a}{b}$$

$$x=\left(\frac{2a}{b}\right)^{\frac{1}{6}}$$

Furthermore, $\frac{d^2U}{dx^2}=\frac{d}{dx}(-12ax^{-13}+6bx^{-7})=156ax^{-14}-42bx^{-8}$

$$x=\left(\frac{2a}{b}\right)^{\frac{1}{6}}$$

$$\frac{d^2U}{dx^2}=\frac{156a}{(\frac{2a}{b})^{\frac{14}{6}}}-\frac{42b}{(\frac{2a}{b})^{\frac{8}{6}}}=156a(\frac{2a}{b})^{-\frac{7}{3}}-42b(\frac{2a}{b})^{-\frac{4}{3}}$$

$$\frac{d^2U}{dx^2}=(\frac{2a}{b})^{-\frac{4}{3}}\{156a(\frac{2a}{b})^{-1}-42b\}$$

$$\frac{d^2U}{dx^2}=(\frac{2a}{b})^{-\frac{4}{3}}\{(156a\times\frac{b}{2a})-42b\}$$

$$\frac{d^2U}{dx^2}=(\frac{2a}{b})^{-\frac{4}{3}}\{36b\}$$

Here, as and both are positive, so the above expression is always positive.

For the obtained value of x, the double differentiation of U is always positive. This ensures that, for this value of x, the system remains in stable equilibrium.