MECHANICAL ENERGY

Mechanical Energy

The total mechanical energy, also called the sum of kinetic energy and potential energy.

$$K + U = ME$$

According to the work – energy theorem,

The total work done by all forces acting on an object equals the change in its kinetic energy.

$$W_{all} = \Delta K$$
$$W_{internal} + W_{external} = \Delta K$$
$$(W_{conservative} + W_{non\text{-}conservative}) + W_{external} = \Delta K$$

If non-conservative forces are absent or if the work done by them is zero, then

$$W_{conservative} + W_{external} = \Delta K$$
$$W_{external} = -W_{conservative} + \Delta K$$
$$W_{external} = \Delta U + \Delta K$$

($\because \Delta U =$ Change in potential energy $= -W_{conservative}$)

$$W_{external} = \Delta(U + K) = \Delta E$$

Here, E is the mechanical energy.

If work done by external force is also zero, then mechanical energy remains constant.

$$W_{external} = 0,$$
$$\Delta E = 0$$

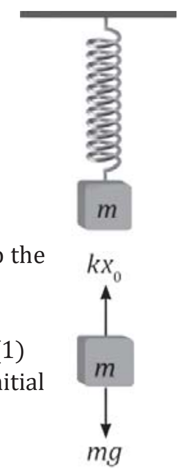
$E =$ Constant

$$K_i + U_i = K_f + U_f$$

Note: If the internal forces are conservative, then the work done by external forces equals the change in mechanical energy.

If no external forces are acting (or if the work done by them is zero) and the internal forces are conservative, the mechanical energy of the system remains constant.

Ex. A block of a mass m is suspended through a spring of spring constant K and is in equilibrium. A sharp blow give the block an initial downward velocity v. how far below the equilibrium position dose the block come to an instantaneous rest?



Sol. Given,

Mass of the block = m

Spring constant = K

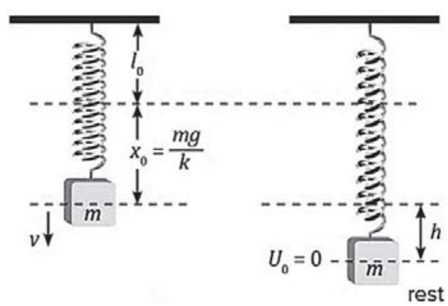
Initial downward velocity = v

Let the original length of the spring be l_0 and extension (at equilibrium) due to the mass be x_0.

At equilibrium before the block start motion.

$$Kx_0 = mg \quad \dots (1)$$

Let h be the maximum extension from the equilibrium position due to the initial below.



Applying the work-energy theorem we get

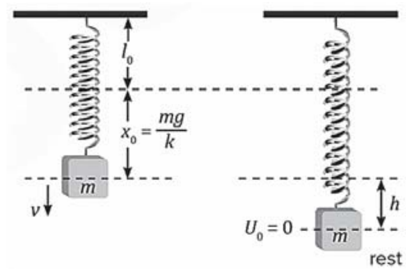
$$W_{all} = \Delta K$$
$$W_g + W_{spring} = K_f - K_i = \frac{1}{2}m(0^2 - v^2) = -\frac{1}{2}mv^2$$
$$mgh - \frac{1}{2}k((x_0 + h)^2 - x_0^2) = -\frac{1}{2}mv^2$$
$$mgh - \frac{1}{2}k(x_0 + h)^2 + \frac{1}{2}kx_0^2 = -\frac{1}{2}mv^2$$

$$\frac{1}{2}kx_0^2 + \frac{1}{2}mv^2 = \frac{1}{2}k(x_0 + h)^2 - mgh$$
$$\frac{1}{2}kx_0^2 + \frac{1}{2}mv^2 = \frac{1}{2}k(x_0^2 + h^2 + 2hx_0) - mgh$$
$$\frac{1}{2}mv^2 = \frac{1}{2}kh^2 + khx_0 - mgh$$

From equation (i), $$\frac{1}{2}mv^2 = \frac{1}{2}kh^2 + mgh - mgh$$
$$\frac{1}{2}mv^2 = \frac{1}{2}kh^2$$
$$h = v\sqrt{\frac{m}{k}}$$

Alternative way

Let the potential energy datum be the lowest point as shown in the figure.



Applying the principal of conservation of mechanical energy we get.

$$K_i + U_i = K_f + U_f$$
$$\frac{1}{2}mv^2 + (mgh + \frac{1}{2}kx_0^2) = 0 + \frac{1}{2}k(h + x_0)^2$$
$$\frac{1}{2}mv^2 + \frac{1}{2}kx_0^2 = \frac{1}{2}k(h^2 + x_0^2 + 2hx_0) - mgh$$
$$\frac{1}{2}mv^2 + \frac{1}{2}kx_0^2 = \frac{1}{2}kh^2 + \frac{1}{2}kx_0^2 + hkx_0 - mgh$$
$$\frac{1}{2}mv^2 = \frac{1}{2}kh^2 + mgh - mgh$$
$$\frac{1}{2}mv^2 = \frac{1}{2}kh^2$$
$$h = v\sqrt{\frac{m}{k}}$$

Ex. The potential energy of a particle of mass 1 Kg free to move along the x-axis is given as follows.

$$U(x) = (\frac{x^2}{2} - x)J$$

If the total mechanical energy of the particle is 2 J, then find the maximum speed of the particle (assuming only conservative force acts on the particle)

Sol. Given,

Mass of the particle $= 1$ Kg

Potential energy, $U(x) = \left(\frac{x^2}{2} - x\right)J$

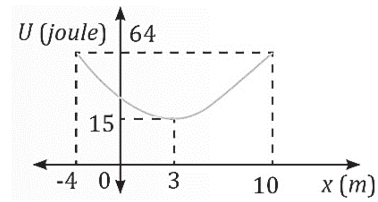
Total mechanical energy $= 2J$

Let the speed of the particle be v.

We know, $$E = K + U$$
$$K + \left(\frac{x^2}{2} - x\right) = 2$$
$$K = 2 + x - \frac{x^2}{2} = \frac{1}{2}v^2 \times 1$$
$$v^2 = 4 + 2x - x^2$$

For v to be the maximum, $$\frac{dv}{dx} = 0 \Rightarrow \frac{d}{dx}(v^2) = 0$$
$$\frac{d}{dx}(4 + 2x - x^2) = 0$$
$$2 - 2x = 0$$
$$x = 1$$
$$v_{max}^2 = 4 + (2 \times 1) - 1^2 = 5$$
$$v_{max} = \sqrt{5}ms^{-1}$$

Ex. A single conservative force, F(x) acts on a particle that moves along the x-axis. The graph of the potential energy with x is given. At x = 5m, the particle has a kinetic energy of 50 f and its potential energy is related to position x as, U = {15 + (x–3)^2} j, where x is in meter.



(a) Mechanical energy of the system
(b) Whether the kinetic energy is maximum of minimum at x =3m.
(c) Maximum value of the kinetic energy

Sol. Given,
At x = 5 m, $K_5 = 50$ J
Potential energy, $U = 15 + (x - 3)^2$J

(a) At x = 5 m, potential energy is given as follows.

$$U_5 = 15 + (5 - 3)^2 \text{ J} = 19 \text{ J}$$

As mechanical energy of the system (E) = Kinetic energy (K) + Potential energy (U)

$$E = 50 + 19 = 69 \text{ J}$$

(b) As only conservative force is applied on the particle, the mechanical energy is constant.

$$K + U = E = \text{ Constant}$$

However from the graph, at x = 3m, the potential energy (U) is minimum. Thus the kinetic energy (K) will be maximum.

(c) We have total mechanical energy of particle, E = 69 J
At x = 3m, U = 15 J, (From the graph)

$$K = E - U = 69 - 15 = 54\text{J}$$