Chapter 8

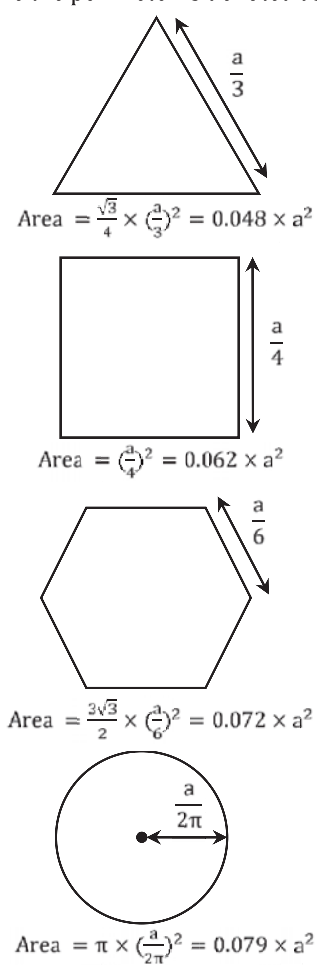
Circular Motion

INTRODUCTION

A circle may be viewed as a polygon possessing an infinite number of sides.

With a constant perimeter, the area of a regular polygon grows as the number of sides increases.

To comprehend the statement above, examine the following scenarios where the perimeter is denoted as 'a':



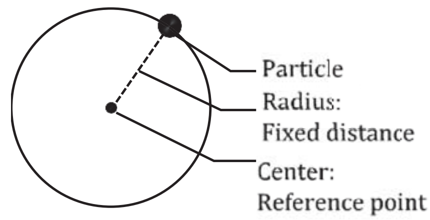
It is evident that as the number of sides in a regular polygon with perimeter 'a' increases, the area also increases, reaching its maximum when the polygon approaches a circle with an infinite number of sides.

- Radius of Curvature and Horizontal Circular Turnings
  1. Radius of Curvature
  Turning on Circular Roads
- Banking of Tracks
  1. Turning on Roads 2. Banking of Roads 3. Death well
- Centrifugal Force and Concept of Apparent Weight
  1. Centrifugal Force
  2. Concept of Apparent Weight

Circular Motion

If a particle traverses a plane while maintaining a constant distance from the reference point, its path or locus is termed a circle. This motion is referred to as circular motion.

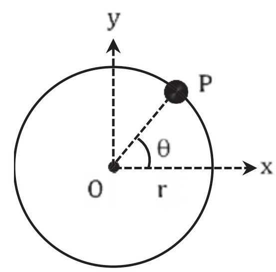
The point used as a reference for measuring the distance of a particle is called the center. The consistent distance from the center to the particle is referred to as the radius.



Circular motion does not require a stationary center; the center can move within the same plane.

Ex. A slender ring (circle) moving along a straight path on the floor exhibits circular motion.

Angular Position (θ):



The angle formed by the position vector in relation to a fixed axis (in this case, the x-axis) is termed angular position.
This is a scalar quantity.
SI unit: rad

Sign convention for angular position

1. If a positive orientation is assigned to clockwise, then counterclockwise will be regarded as negative.

2. If a negative orientation is assigned to clockwise, then counterclockwise will be considered positive.

Unit conversion of degree to radian:

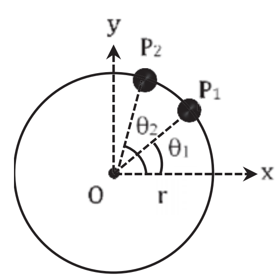
Angle = Ratio of Arc to Radius

$$\text{For full circle, Angle } = \frac{\text{Perimeter}}{\text{Radius}} = \frac{2 \times \pi \times r}{r} = 2\pi \text{ or } 360^\circ$$

$$360^\circ = 2\pi\text{rad}$$

$$1^\circ = \frac{2\pi}{360}\text{rad}$$

Angular Displacement (Δθ)



The angular displacement is defined as the angle by which the position vector of a moving particle rotates within a specified time interval with respect to a designated origin and reference axis.
This is a scalar quantity.
SI unit: rad
Imagine a particle in circular motion transitioning from point P_1, situated at an angle $+\theta_1$ counterclockwise from the horizontal axis, to point P_2, located at an angle $+\theta_2$ counterclockwise from the horizontal axis.
Angular displacement, $\Delta\theta = \theta_2 - \theta_1$

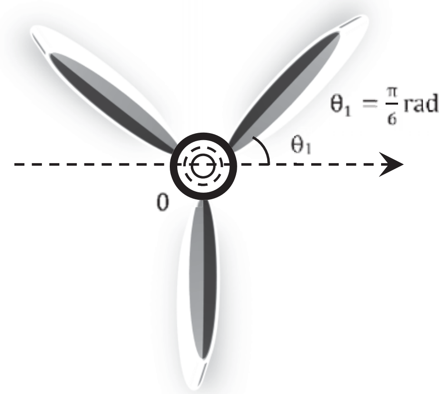
To establish angular position in circular motion, the following are necessary:

1. Centre

2. Reference axis

3. Sense of rotation (Clockwise or Anticlockwise)

Ex. The angular orientation of one of the blades of a fan is $\theta_1 = \frac{\pi}{6}$ rad in relation to the axis as depicted in the figure. If the fan is rapidly turned on and off, the blade's angular position undergoes change to $\theta_2 = \frac{\pi}{3}$ rad what is the angular displacement for each of the blades?



Sol. Given, $\theta_1 = +\frac{\pi}{6}$

(Taking counter - clockwise as positive direction)

$$\theta_2 = +\frac{\pi}{3}$$

Now, angular displacement

$$\Delta\theta = \theta_2 - \theta_1 = \frac{\pi}{3} - \frac{\pi}{6}$$

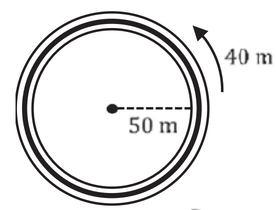
Angular Displacement, $\Delta\theta = \frac{\pi}{6}$ rad (counter-clockwise)

Presently, all the blades rotate at an identical angular speed. Consequently, the angular displacement of all blades is uniform.

Ex. A student completes a 40 m run around a circular track with a radius of 50 m. What is the angular displacement?

(A) 0.6 rad (B) 0.8 rad

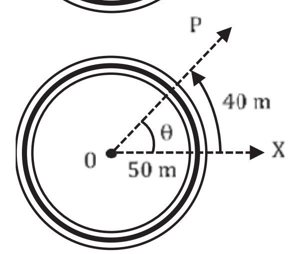
(C) 1.0 rad (D) 1.2 rad



Sol. Upon covering a distance of 40 m, the student arrives at point P at an angle θ, as illustrated in the figure. The angular displacement is.

$$\Delta\theta = \theta_2 - \theta_1$$
$$\theta_1 = 0 \text{ and}$$
$$\theta_2 = \theta$$

Now, $\theta = \frac{\text{Arc}}{\text{Radius}}\text{rad} = \frac{40}{50}\text{rad}$



Angular Displacement

$= \Delta\theta = 0.8$rad (counter-clockwise)

Thus, option (B) is the correct answer.

1. The angular displacement for a full circle is 2π.
2. Angular displacements are combined or subtracted according to the direction of rotation.

Average Angular Velocity (Δω)

The quotient of the total angular displacement divided by the total time taken is referred to as average angular velocity. This is a scalar quantity.

SI unit: rads^{-1}

Let θ_1 and θ_2 be the angular positions at time t_1 and t_2, respectively. Then,

$$\text{Average angular velocity} = \frac{\text{Net angular displacement}}{\text{Total time taken}}$$

$$\Delta\omega = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

Ex. If a marathon runner covers half the distance of a 5 km circular track in 20 minutes and the remaining distance in 30 minutes, what is the average angular velocity?

Sol. The runner finishes the entire circular track in 20 + 30 = 50 min

Angular displacement of runner = Angular displacement of one circle = 2π

So, average angular velocity,

$$\omega_{avg} = \frac{2\pi}{50 \times 60}\text{rads}^{-1} = \frac{\pi}{1500}\text{rads}^{-1}$$

Ex. A diver completes 2.5 revolutions during the descent from a 10 m platform to the water below. Assuming no initial vertical velocity, determine the average angular velocity for this dive. [Using $g = 10\ ms^{-2}$]

Sol. Given,

Number of revolutions = 2.5

Height of the platform = 10 m Angular displacement in 2.5 revolutions is,

$$\Delta\theta = 2.5 \times 2\pi = 5\pi rad$$

Now, to determine the time, we will apply equations of motion since the diver is in free fall.

Taking downward direction as positive.

Initial velocity (u) = $0\ ms^{-1}$

Acceleration = $+g\ ms^{-2}$

Displacement (s) = 10 m Now,

We know that,

$$s = ut + \frac{1}{2} \times gt^2 \quad \dots (1)$$

$$u = 0ms^{-1}$$

$$g = 10ms^{-2}$$

$$s = 10m$$

Put in equation (1),

$$10 = 0 \times t + \frac{1}{2} \times 10 \times t^2$$

$$10 = 5t^2$$

$$t = \sqrt{2}s$$

$$\omega_{avg} = \frac{\Delta\theta}{\Delta t} = \frac{5\pi}{\sqrt{2}} rads$$

PARAMETERS OF CIRCULAR MOTION

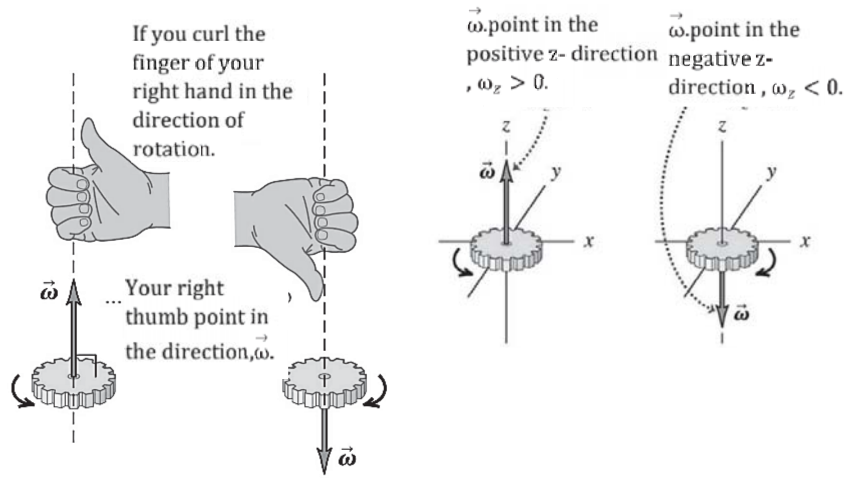
1. Instantaneous Angular Velocity:

The variable "t" represents the limit of average velocity as Δt approaches zero, that is,

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\theta}}{\Delta t} = \frac{d\vec{\theta}}{dt}$$

Since infinitesimally small angular displacement $d\vec{\theta}$ is a vector quantity, instantaneous angular velocity $\vec{\omega}$ is also a vector, whose direction is given by right hand thumb rule.

Right hand thumb rule:



- Angular velocity has dimension of $[T^{-1}]$ and Si unit rad/s.
- For a rigid body, as all points will rotate through same angle in same time, angular velocity is a characteristic of the body as a whole, e.g. angular velocity of all points of earth about earth`s axis is (2p/24) rad/hr.
- If a body makes 'n' rotation in't' second then average angular velocity in radian per second will be.

$$\omega_{cv} = \frac{2n\pi}{T}$$

- If T is the period and 'f' the frequency of uniform circular motion.

$$\omega_{av} = \frac{2\pi}{T} = 2nf$$

Ex. Is the angular velocity of rotation of hour hand of a watch greater or smaller than the angular velocity of Earth's rotation about its own axis?

Sol. Hour hand completes one rotation in 12 hours while Earth completes one rotation in 24 hours. So, angular velocity of hour hand is double the angular velocity of Earth.

$$(\omega = \frac{2\pi}{T}).$$

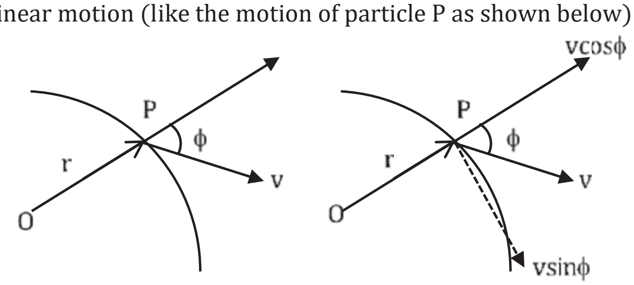
2. Relation between Linear Velocity (V) and angular Velocity (ω):

For a particle undergoing circular motion

$$v = \lim_{\Delta t \to 0} (\frac{\Delta s}{\Delta t}) = r \lim_{\Delta t \to 0} (\frac{\Delta \theta}{\Delta t}) \Rightarrow V = r\omega$$

Thus different points of second`s hand of a clock are rotating with same angular velocity but with different speed and its tip has greatest speed.

For any curvilinear motion (like the motion of particle P as shown below)



If the particle has only velocity component tcos ϕ(along $\vec{r}$) need not turn his head to always look at the particle i.e., this component does not contribute in angular motion. Thus only the component vsin ϕ responsible for changing the angular displacement.

$$\omega = \frac{v\sin\phi}{r}$$

$$\text{In general, } \omega = \frac{\text{velocity component perpendicular to the line joining the particle and the observer}}{\text{distance between the particle and observer}}$$

$$\omega = \frac{v_\perp}{r}$$

Ex. A fan is rotating with an angular velocity of 100 rev/sec. Upon switching off, it takes 5 minutes to come to a halt.

(a) Determine the total number of revolutions made before it stops. (Assume uniform angular retardation)

(b) Calculate the value of angular retardation.

(c) Find the average angular velocity during this interval.

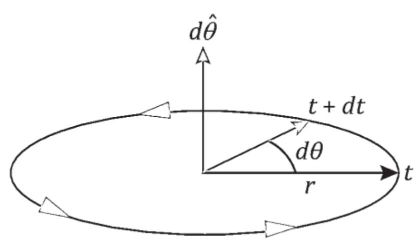
Sol. (a) $\theta = (\frac{\omega+\omega_0}{2})t = (\frac{100+0}{2}) \times 5 \times 60 = 15000$ revolution

(b) $\omega = \omega_0 + \alpha t \Rightarrow 0 = 100 - \alpha(5 \times 60) \Rightarrow \alpha = \frac{1}{3}\text{rev/sec}^2$

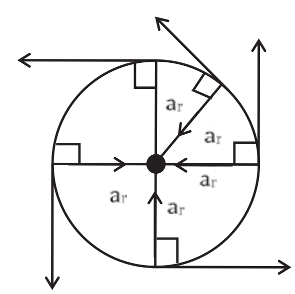
(c) $\omega_{av} = \frac{\text{Total Angle of Rotation}}{\text{Total time taken}} = \frac{15000}{50\times60} = 50\text{rev./sec.}$

Axial Vectors

Infinitesimal angular displacement and infinitesimal angular velocity are axial vectors, indicating that their direction aligns with the axis of rotation of the particle and is perpendicular to the plane of rotation.



Uniform circular motion



When a particle is moving with constant speed in circular path, its motion is called uniform circular motion. Although magnitude of velocity is constant but its direction, changes continuously. It means it is continuously having some acceleration. This acceleration is always directing towards the centre. This is called acceleration (a_r) or centripetal acceleration (a_c).

$$\vec{r} = r(\cos\theta i + \sin\theta \hat{j})$$

$$\vec{v} = -v\sin\theta i + v\cos\theta \hat{j} = v(-\sin\theta i + \cos\theta \hat{j})$$

$$\vec{a} = \frac{d\vec{v}}{dt} = v(-\cos\theta i - \sin\theta j)(\frac{d\theta}{dt}) + v(\sin\theta i + \cos\theta j)(\frac{dv}{dt})$$

$$|\vec{V}| \text{ is constant } \Rightarrow \frac{d|\vec{V}|}{dt} = 0 \text{ i.e., } \frac{dv}{dt} = 0$$

$$\frac{d\theta}{dt} = \omega$$

$$\vec{a} = -v\omega(\cos\theta \hat{i} + \sin\theta \hat{j}) \Rightarrow \vec{a} = v\omega\hat{a}$$

$$\hat{a} = -(\cos\theta i + \sin\theta j) = \hat{r}$$

Thus direction of this acceleration is opposite to i.e., radially inwards.

$$|\vec{a}| = v\omega \Rightarrow a_r = v\omega \text{ or } \frac{v^2}{r} \text{ or } \omega^2 r[\because v = \omega r]$$

Magnitude of this acceleration is constant but direction changes continuously, always being normal to velocity as shown in the figure.