A TALE OF ACCELERATIONS

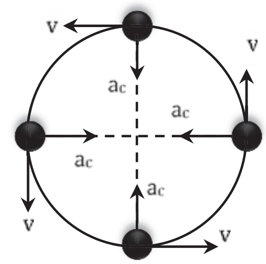
Classification of Acceleration

Acceleration can be categorized in the following ways:

1. Depending on time interval
   (a) Average acceleration (b) Instantaneous acceleration
2. Depending on motion
   (a) Linear acceleration (b) Angular acceleration

Type of acceleration	Average acceleration	Instantaneous acceleration
Linear	$\vec{a}_{avg} = \frac{\Delta\vec{V}}{\Delta t}$	$\vec{a} = \frac{d\vec{V}}{dt}$
Angular	$\alpha_{avg} = \frac{\Delta\omega}{\Delta t}$	$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$

Centripetal Acceleration (Quantitative Analysis)



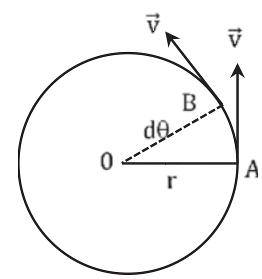
Contemplate a particle traveling in a circle with a radius r at a consistent speed v, as depicted in the figure.

In Uniform circular motion,

Total acceleration = Centripetal acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \vec{a}_c$$

$$|\vec{a}_c| = \frac{|d\vec{v}|}{dt}$$



The particle moves from point A to point B with an angular displacement dθ within the time interval dt.

Because the speed is constant, the velocity magnitude at A equals the velocity magnitude at B.

$$|\vec{V}_A| = |\vec{V}_B| = V$$

Change in velocity,

$$d\vec{V} = \vec{V}_B - \vec{V}_A$$

$$|d\vec{V}| = |\vec{V}_B - \vec{V}_A| = \sqrt{|\vec{V}_B|^2 + |\vec{V}_A|^2 - 2 \times |\vec{V}_A| \times |\vec{V}_B|\cos(d\theta)}$$

$$|d\vec{V}| = \sqrt{V^2 + V^2 - 2V^2\cos(d\theta)}$$

$$d\vec{V}| = \sqrt{2v^2(1-\cos(d\theta))} = \sqrt{2v^2(2\sin^2(\frac{d\theta}{2}))}; (\because\ (1-\cos\theta) = 2\sin^2(\frac{\theta}{2}))$$

$$d\vec{V}| = 2v\sin(\frac{d\theta}{2})$$

However, when θ is very small, $\sin\theta \sim \theta$, so,

$$\sin(\frac{d\theta}{2}) \approx \frac{d\theta}{2}$$

$$|d\vec{v}| = vd\theta$$

Now, acceleration is,

$$\vec{a} = \frac{d\vec{V}}{dt} = \vec{a}_c$$

$$|\vec{a}_c| = \frac{|d\vec{v}|}{dt}$$

$$\left|\vec{a}_c\right| = \frac{Vd\theta}{dt} = V\omega \qquad \dots (1)$$

$$\vec{a}_c = \vec{\omega} \times \vec{V}$$

$$V = \omega r$$

$$a_c = \omega r \times \omega = \omega^2 r \qquad \text{... (2)}$$

$$a_c = V \times \frac{V}{r} = \frac{v^2}{r} \qquad \text{... (3)}$$

Moreover, it is understood that the centripetal acceleration's direction aligns with the inward radius vector.

$$\vec{a}_c = -V\omega\hat{r} = -\frac{V^2}{r}\hat{r} = -\omega^2 r\hat{r}; \text{ (where } \hat{r} \text{ is the unit vector along the radius)}$$

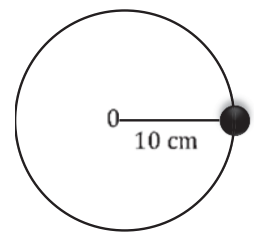
Scalar relation	Vector relation
$dl = r d\theta$	$d\vec{l} = r d\vec{\theta}$
$V = \omega r$	$\vec{V} = \vec{\omega} \times \vec{r}$
$a_t = \alpha r$	$\vec{a}_t = \vec{\alpha} \times \vec{r}$
$a_c = \omega V$	$\vec{a}_c = \vec{\omega} \times \vec{V}$

Ex. Determine the magnitude of the centripetal acceleration for a particle moving in a circle with a radius of 10 cm, maintaining a uniform speed and completing one full circle in 4 seconds.

Sol. Given, Radius (r) = 10 cm

Time to complete one rotation (t) = 4 s

The speed remains constant, indicating a uniform circular motion. Let the speed be v, and let a_c represent the magnitude of the centripetal acceleration .Now



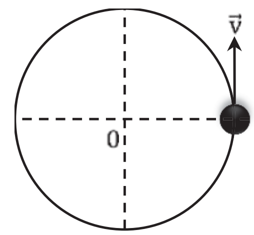
$$a_c = \frac{v^2}{r} = \frac{\left(\frac{2\pi r}{t}\right)^2}{r}$$

$$a_c = \frac{\left(\frac{2\pi \times 10}{4}\right)^2}{10} = 2.5\pi^2 \text{ cms}^{-2}$$

Total Acceleration in Circular Motion

Contemplate a particle engaged in circular motion.

At any given moment t, consider the particle's velocity as $\vec{v}$, and the overall acceleration as $\vec{a}$.



$$\vec{a} = \frac{d\vec{v}}{dt}$$

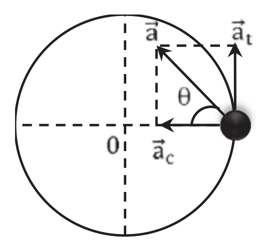
$$\vec{V} = \vec{\omega} \times \vec{r}$$

$$\vec{a} = \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

$$\vec{a} = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}; 1(\because (fg)' = f'g + fg')$$

$$\vec{a} = \alpha \times \vec{r} + \vec{\omega} \times \vec{V}$$

$$\vec{a} = \vec{a}_t + \vec{a}_c$$



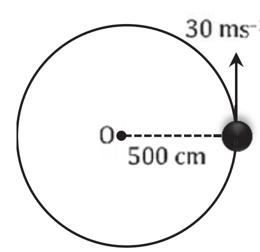
The total acceleration is the combined vector sum of centripetal acceleration and tangential acceleration.

$$\vec{a} = \vec{a}_t + \vec{a}_c$$

$$|\vec{a}| = \sqrt{a_t^2 + a_c^2}$$

$$\tan\theta = \frac{a_t}{a_c}$$

Ex. A particle is orbiting in a circular path with a radius of 500 m at a speed of 30 m/s. Its speed is increasing at a rate of 2 ms^2. What is its acceleration?



Sol. Given,

Radius (r) = 500 m

Initial speed (u) = 30 ms^{-1}

Tangential acceleration (at) = 2 ms^{-2}

Let the speed be v and the magnitude of centripetal acceleration be ac .We know,

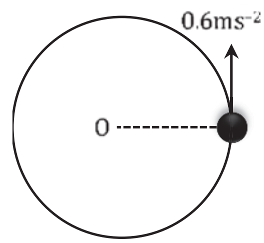
$$a_c = \frac{V^2}{r} = \frac{30^2}{500} = \frac{9}{5}\ ms^{-2}$$

Total acceleration,

$$a = \sqrt{a_t^2 + a_c^2} = \sqrt{2^2 + \frac{9^2}{5^2}}$$

$$a = \sqrt{\frac{181}{25}} ms^{-2}$$

Ex. A particle initiates motion with a constant magnitude tangential acceleration of 0.6 ms^{-2} along a circular path. If it starts slipping when its total acceleration reaches 1 ms^2, determine the angle it would have rotated before slipping occurs.



Sol. Given, Initial velocity = 0 ms^{-1}

Tangential acceleration (a_t) = 0.6 ms^{-2}

Let the particle begin to slip at time t

$$\text{At } t = ts, a = 1ms^{-2}$$

Consider the angular displacement before slipping as $\Delta\theta$.

$$a = \sqrt{a_t^2 + a_c^2}$$

$$1 = \sqrt{0.6^2 + a_c^2}$$

Squaring both sides,

$$1 = 0.36 + a_c^2$$

$$a_c^2 = 1 - 0.36 = 0.64$$

$$a_c = 0.8ms^{-2}$$

For constant tangential acceleration, we have

$\omega^2 = \omega_o^2 + 2\alpha\Delta\theta$; (where, $\alpha =$ Angular acceleration and $\Delta\theta =$ Angular displacement)

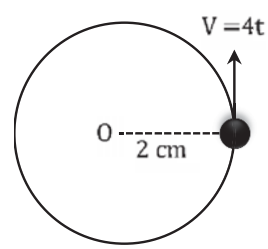
$$\omega^2 = 0^2 + (2 \times \frac{a_t}{r} \times \Delta\theta)$$

$$\Delta\theta = \frac{\omega^2 r}{2a_t} = \frac{0.8}{2 \times 0.6} = \frac{2}{3}\text{rad}$$

Ex. A particle is moving in a circle of radius 2 cm at a speed given by v = 4t, where v is in cm s-1 and t is in seconds.

(a) Find the tangential acceleration at t = 1 s

(b)Find the total acceleration at t = 1 s



Sol. Given, Speed (v) = 4t

Radius (r) = 2 cm

Consider the magnitude of tangential acceleration as "at" and the magnitude of centripetal acceleration as "a_c."

We know,

$$a_t = \frac{dv}{dt} = \frac{d(4t)}{dt}$$

$$a_t = 4\ cms^{-2}$$

$$\text{At } t = 1s,$$

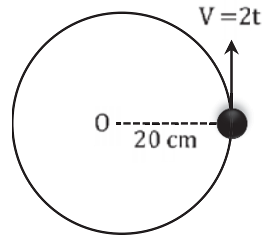
$$V = 4 \times 1 = 4\ cms^{-1}$$

$$a_c = \frac{v^2}{r} = \frac{4^2}{2} = 8\ cms^{-2}$$

So, total acceleration at t = 1s.

$$a = \sqrt{a_t^2 + a_c^2} = \sqrt{4^2 + 8^2} = 4\sqrt{5}\ cm\ s^{-2}$$

Ex. A particle moves in a circle of radius 20 cm. Its linear speed is given by v = 2t, where t is in seconds and v in ms^{-1}. Find the radial and tangential acceleration at t = 3 s.



Sol. Given, Speed (v) = 2t

Radius (r) = 20 cm = 0.2 m

Consider at as the magnitude of tangential acceleration and a_c as the magnitude of centripetal acceleration.

We know

$$a_t = \frac{dv}{dt} = \frac{d(2t)}{dt}$$
$$a_t = 2ms^{-2}$$
$$\text{At } t = s,$$
$$V = 2 \times 3 = 6ms^{-1}$$
$$a_c = \frac{v^2}{r} = \frac{6^2}{0.2} = 180ms^{-2}$$