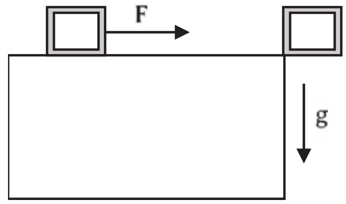
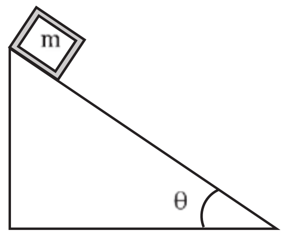
CONSTRAINED MOTION: PULLEY BLOCK SYSTEM & APPLICATION OF STRING CONSTRAINTS

Introduction to Constrained Motion

Constrained motion occurs when an object is compelled to travel within defined limitations. In this context, the object is constrained to travel along the surface of the block due to applied force and subsequently descends vertically due to the influence of gravity.



Similarly, in this case, the block is restricted to slide along an inclined plane. The force responsible for imposing this constraint on the body is referred to as the constraint force.



There are two primary constraints:

1. String/Rod constraints
2. Wedge constraints

String Constraints

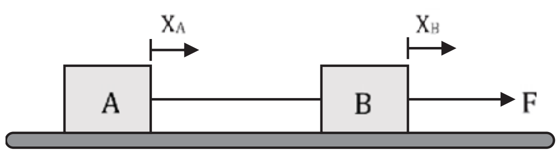
Contemplate objects linked by a string possessing the following characteristics:

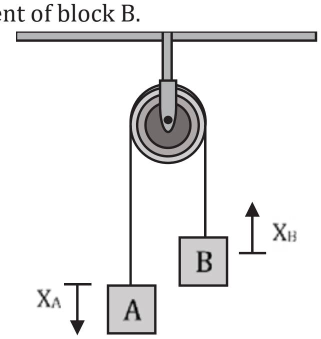
The string maintains a constant length, signifying that it is inextensible. This implies that if one block moves a distance of x, the other block also moves the same distance.

This relationship can be expressed mathematically as:

$$X_A = X_B$$

Where X_A, is displacement of block A and X_B is displacement of block B.





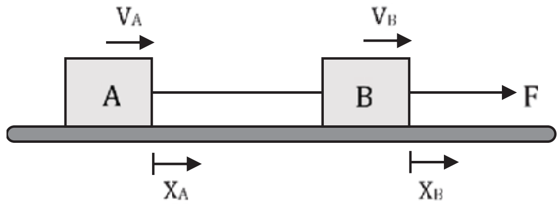
It consistently remains taut and does not become slack.

The motion characteristics of these objects along the length of the string and in the direction of extension possess a specific relationship among their parameters. In this context, the parameters of motion refer to displacement, velocity, and acceleration, and the objects denote the bodies directly connected to the string.

A definite relationship refers to the constraint relation, such as $X_A = X_B$

Constraint Relations

Mathematical relationships that describe the motion of two or more bodies within a system are referred to as constraint relations.



In the provided system, given that the length of the string remains constant,

$$X_A = X_B$$

Taking the derivative of the equation with respect to time.

$$\frac{dx_A}{dt} = \frac{dx_B}{dt}$$

It is known that

$$\frac{dx_A}{dt} = V_A \text{ and } \frac{dx_B}{dt} = V_B$$

$$V_A = V_B$$

Taking the second derivative of the equation with respect to time,

$$\frac{dv_A}{dt} = \frac{dv_B}{dt}$$

However, it is known that

$$\frac{dv_A}{dt} = a_A \text{ and } \frac{dv_B}{dt} = a_B$$

$$a_A = a_B$$

Hence, we can establish a connection between the parameters by utilizing the inextensibility constraint of the string.

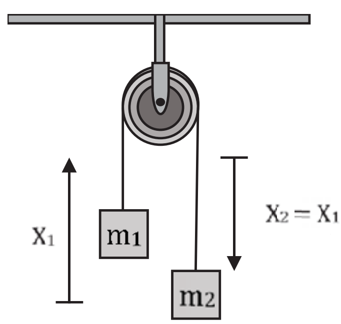
Therefore, the constraint relations in this case are $x_A = x_B, v_A = v_B$ and $a_A = a_B$

Pulley-Block System

Atwood's machine

Atwood's machine refers to a contraption where two objects are linked by a string passing over a pulley in such a way that if one descends, the other ascends.

In this setup, heavier objects will descend, while lighter objects will ascend.



Examine the magnitude of displacement for two objects. According to the inextensibility constraint of the string,$x_1 = x_2$

Likewise, since they are covering an equal distance within the same time interval,

$$v_1 = v_2$$

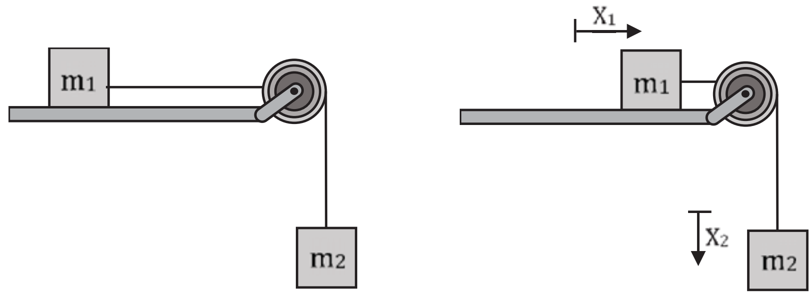
$$a_1 = a_2$$

Example In the case of the depicted blocks, the string is inextensible. Examine the magnitude of displacement for the two objects.$x_1 = x_2$

Similarly, since they are traversing an equal distance within the same time interval,

$$v_1 = v_2$$

$$a_1 = a_2$$

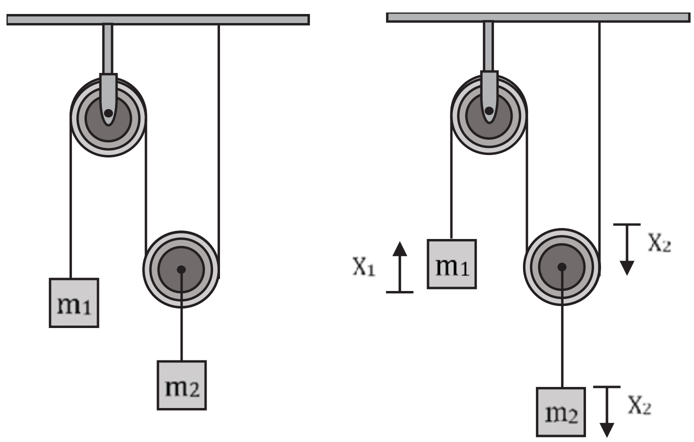


Hence, we can represent the system in a mathematical equation form.

Ex. **System with movable pulley**

Recognize all the objects and the number of strings involved in the problem. In this scenario, the distance traveled by the movable pulley is identical to the distance covered by block 2 since they are connected by a single inextensible string.

While the movable pulley descends by the distance x_2, the strings on both sides of the pulley must also move downward by the same distance, namely x_2. Therefore, lowering the movable pulley by the distance x_2 requires a length of $2x_2$ from the string. This length is entirely provided by the left side of the string, as the total length remains constant.



This establishes a correlation between x_1 and x_2.

Thus, $x_1 = 2x_2$

Similarly, $v_1 = 2v_2$

And $a_1 = 2a_2$

Steps for writing string constraint relation

Step: 1 Recognize all the elements and count the number of strings in the given scenario. The depicted system comprises two objects and two strings, as illustrated in the figure.

Step: 2 Establish variables and orientations to symbolize the motion parameters, including displacement, velocity, acceleration, etc. In the system, the two blocks, string, and a pulley exhibit mobility. Assign variables v_1 and v_2 to denote the velocities of block 1 and 2 respectively, and designate v_p as the velocity of the pulley.

Step: 3 Recognize a string, partition it into distinct linear segments, and formulate the constraint equation. In this context, the upper string is segmented into lengths l_1, l_2, and l_3, while the lower string has a length of l_4, as depicted in the figure.

Given that the total length of any inextensible string always remains constant,

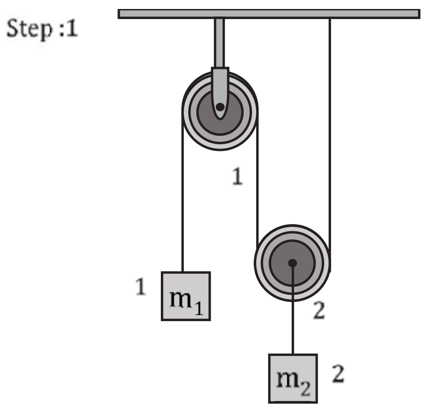
$$I_1 + I_2 + I_3 + I_4 + I_5 + \cdots = I(\text{ Constant })$$

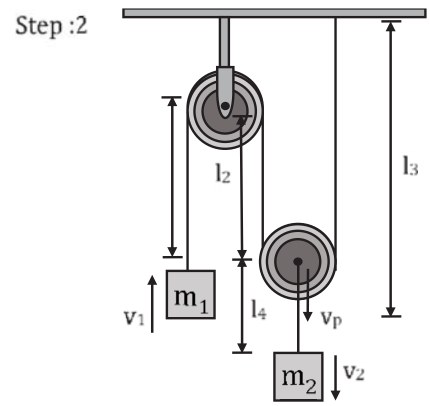
For string 1

$$I_1 + I_2 + I_3 = \text{ Constant}$$

For string 2

$$I_4 = \text{ Constant}$$





Step: 4 Take the derivative with respect to time.

$l_1 + I_2 + \cdots = I(\text{ Constant })$

$$\frac{dl_1}{dt} + \frac{dl_2}{dt} + \frac{dl_3}{dt} = 0$$

$$i_1 + i_2 + i_3 = 0$$

$\frac{dl_1}{dt} =$ rate of increment of intercept 1

Rather than repeatedly using the differentiation symbol, we can express it more succinctly as,

$$\frac{dl_1}{dt} = \dot{I}_1$$

A single dot indicates the differentiation of a quantity with respect to time, while two dots signify the double differentiation of a quantity with respect to time.

Sign convention is adhered to base on whether there is an increase or decrease in length, as follows:

Positive sign ⇒ increase in length with time Negative sign ⇒ decrease in length with time

Observing the figure, it is evident that the length l1 decreases over time at a rate equal to the velocity of block 1 (v_1), hence it is considered negative. On the other hand, both l_2 and l_3 increase at a rate equal to the velocity of the pulley (v_p), so they are considered positive.

Step: 5 Express differentials in relation to velocities and resolve the resulting equations.

$$i_1 + i_2 + i_3 = 0$$

$$(-V_1) + V_p + V_p = 0$$

$$v_1 = 2v_p \quad \ldots (1)$$

The velocity of intercepts l_2 and l_3 equals the velocity of the movable pulley.

Step: 6 Execute all the steps separately for each string. Concerning the string with intercept l_4,

It represents the overall length of the second string and therefore remains constant.

$$\dot{I}_4 = 0$$

$$V_2 - V_p = 0$$

$$V_2 = V_p \quad \ldots (2)$$

From (1) and (2),

$$v_1 = 2v_2$$

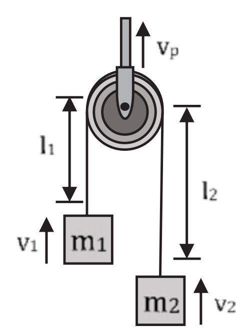
Similarly

$$a_1 = 2a_2$$

$$x_1 = 2x_2$$

Velocity and Acceleration of Moving Pulley

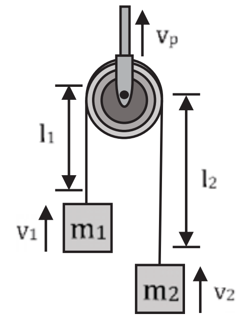
We can find the restricted connection for the moving pulley using these vector equations in a different way.



$$\vec{V}_p = \frac{\vec{V}_1 + \vec{V}_2}{2}$$

$$\vec{a}_p = \frac{\vec{a}_1 + \vec{a}_2}{2}$$

The provided equation holds true for all situations, regardless of the direction of velocity. To demonstrate this, let's examine all conceivable scenarios for the movable pulley.



Case: 1 Both masses and pulley are going upward

Step: 1 Recognize all the elements and count the number of strings in the scenario. In this case, there are two objects, and the pulley is ascending.

Step: 2 Assign variables and directions to symbolize the parameters of motion, including displacement, velocity, acceleration, etc. In this instance, the pulley is moving in an upward direction.

Step: 3 Recognize a string, partition it into distinct linear segments, and formulate the constraint equation. The sum of l_1 and l_2 equals a constant.

Step 4: Take the derivative with respect to time.

$l_1 + I_2 = I(\text{ Constant })$

$$\frac{dl_1}{dt} + \frac{dl_2}{dt} = 0$$

$$i_1 + i_2 = 0$$

Step: 5 Express differentials in relation to velocities and then solve the resulting equations.

$$i_1 + i_2 = 0$$
$$(v_p - V_1) + (v_p - V_2) = 0$$
$$2v_p - v_1 - v_2 = 0$$
$$2v_p = v_1 + v_2$$
$$v_p = \frac{v_1 + v_2}{2}$$

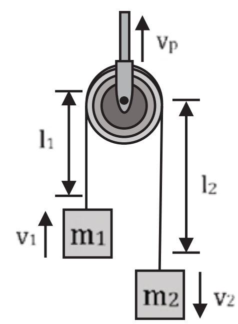
Alternative method

We possess the vector formula,

$$\vec{V}_p = \frac{\vec{V}_1 + \vec{V}_2}{2}$$

Adopting the upward direction as positive, we obtain

$$V_P = \frac{V_1 + V_2}{2}$$



Case: 2 Mass m_1 is going upward and m_2 is going downward

$$i_1 + i_2 = 0$$
$$+(v_p - v_1) + (v_p + v_2) = 0$$
$$2v_p - V_1 + V_2 = 0$$
$$2v_p = v_1 - V_2$$
$$v_p = \frac{v_1 - V_2}{2}$$

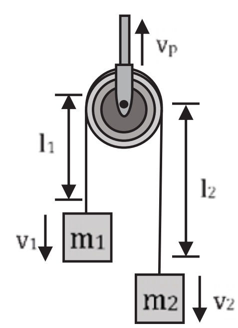
Alternative method

We have the vectorial formula

$$\vec{V}_p = \frac{\vec{V}_1 + \vec{V}_2}{2}$$

By choosing upward direction as positive we get

$$V_P = \frac{V_1 - V_2}{2}$$



Case: 3 Both masses are moving downward and pulley is moving upward

$$i_1 + i_2 = 0$$
$$+(v_p + v_1) + (v_p + v_2) = 0$$
$$2v_p + v_1 + v_2 = 0$$
$$2v_p = -v_1 - v_2$$
$$v_p = \frac{-V_1 - v_2}{2}$$

Alternative method

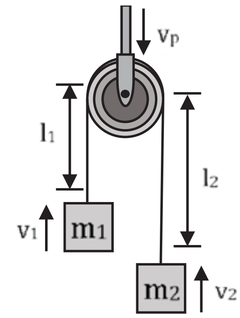
We possess the vector formula.

$$\vec{V}_p = \frac{\vec{V}_1 + \vec{V}_2}{2}$$

By considering the positive direction as upward, we obtain

$$V_P = \frac{-V_1 - V_2}{2}$$

Case: 4 Both masses are moving upward and pulley is moving downward.



$$i_1 + i_2 = 0$$
$$-(v_p + v_1) - (v_p + V_2) = 0$$
$$-2V_p - V_1 - V_2 = 0$$
$$-2V_p = V_1 + V_2$$
$$-V_p = \frac{V_1 + V_2}{2}$$

Alternative method

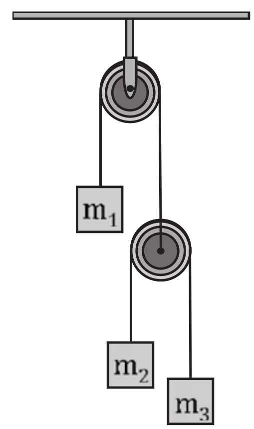
We hold the vector formula.

$$\vec{V}_p = \frac{\vec{V}_1 + \vec{V}_2}{2}$$

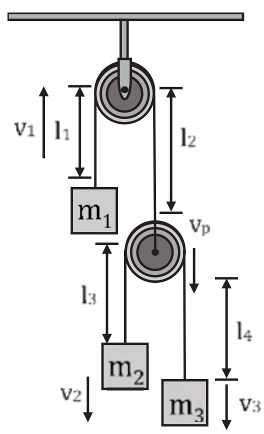
Opting for the positive direction as upward, we obtain,

$$-V_P = \frac{V_1 + V_2}{2}$$

Ex. Determine the constraint relationship among the accelerations of the three blocks.



Sol. In this system, three blocks, two strings, and a movable pulley are present. Segment the strings using intercepts and allocate variables as indicated.



Regarding l_1, the upper point remains fixed, while the lower point moves upward. Consequently, l_1 decreases, and l_2 increases.

$$\dot{l}_1 + \dot{l}_2 = 0$$
$$-v_1 + v_p = 0$$
$$v_1 = v_p \quad \text{... (1)}$$

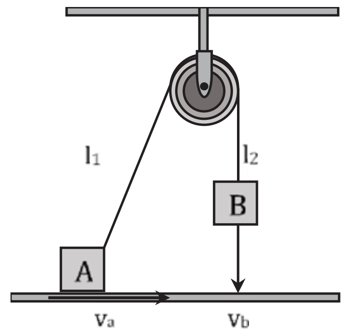
For the second string,

$$\dot{l}_3 + \dot{l}_4 = 0$$
$$+(v_2 - v_p) + (v_3 - v_p) = 0$$
$$v_p = \frac{v_2 + v_3}{2} \quad \text{... (2)}$$

From equation (1) and (2)

$$v_1 = \frac{v_2 + v_3}{2}$$
$$a_1 = \frac{a_2 + a_3}{2}$$

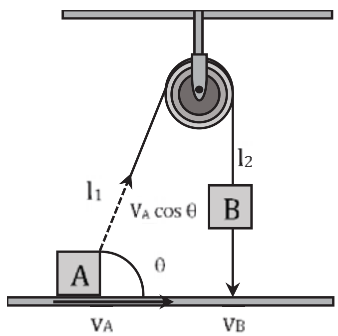
Ex. In the illustrated system, block A is pulled to the right with a speed v_A. Determine the velocity of block B at the moment when the string forms an angle θ with the horizontal.



Sol. As block A approaches, block B will descend. Partition the string into intercepts as depicted.

$l_1 + l_2 =$ Constant

$$\dot{l}_1 + \dot{l}_2 = 0$$
$$-v_A \cos\theta + v_B = 0$$
$$v_B = v_A \cos\theta$$



Alternative way

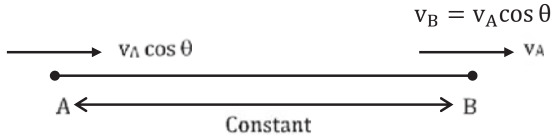
In the context of an inextensible string, the velocity components at the endpoints are identical along the length of the string. Consequently, consider the velocity components at the endpoints.

Velocity component of end B = v_B

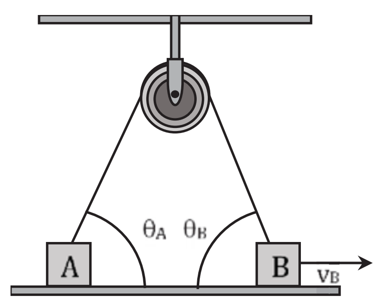
Velocity component of end A along the string = v_B cos θ

Setting these two components equal

$$v_B = v_A \cos\theta$$



Ex. In the depicted system, determine the velocity of block A when B is pulled to the right at the speed v_B, expressed in terms of the angles formed by the string portions with the horizontal at that moment.



Sol. In this scenario, block B moves to the right. Consider the velocity component along the length of the string at both ends.

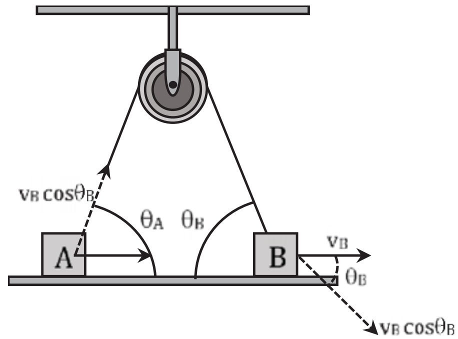
The component of the velocity at end A along the length of the string = $v_A \cos\theta_A$ ... (1)

The component of the velocity at end A along the length of the string = $v_B \cos\theta_B$ ... (2)

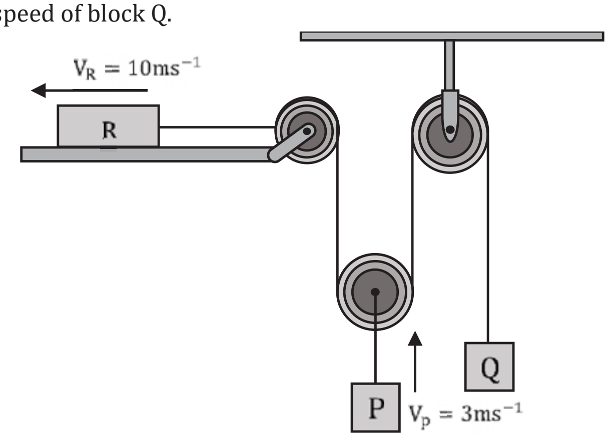
Nevertheless, it is established that the velocity components at the endpoints along a string are equivalent.

Hence, From (1) and (2),

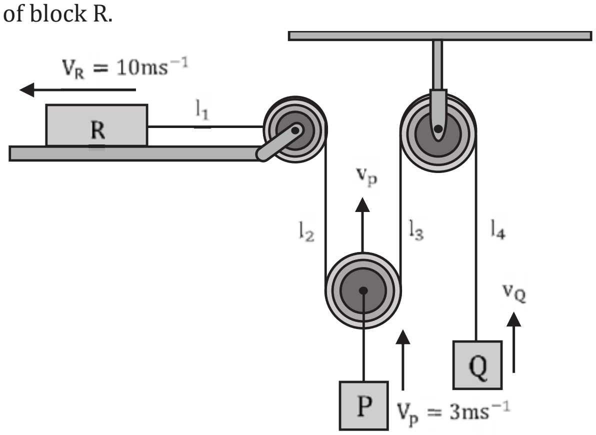
$$v_A \cos\theta_A = v_B \cos\theta_B$$
$$v_A = v_B \frac{\cos\theta_B}{\cos\theta_A}$$



Ex. In the illustrated system, where block R moves to the left at 10 ms^{-1} and block P ascends at 3 ms^{-1}, determine the speed of block Q.



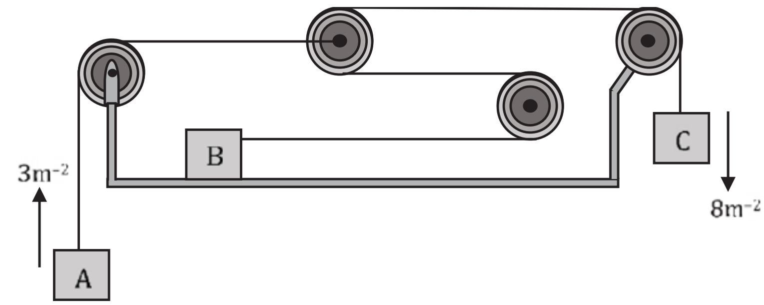
Sol. In this scenario, the direction of motion for block Q is uncertain. For the purpose of analysis, we presume that the block is ascending. If this assumption proves incorrect, a negative sign in the velocity result would indicate that the block is moving in the opposite direction to what was initially assumed. Therefore, we assume that block Q moves upward concurrently with the leftward motion of block R.



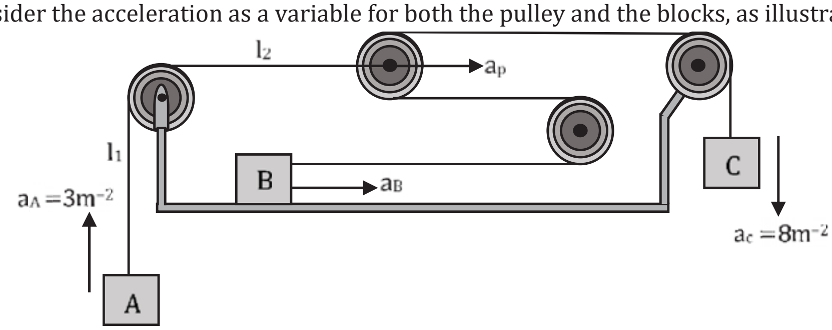
In the system, there are three blocks and a movable pulley.
Partition the string into four intercepts as depicted.

$$\dot{l}_1 + \dot{l}_2 + \dot{l}_3 + \dot{l}_4 = 0$$
$$+v_R - v_p - v_p - v_Q = 0$$
$$v_Q = v_R - 2v_p$$
$$v_Q = 10 - 2(3)$$
$$v_Q = 4ms^{-1}$$

Ex. Determine the acceleration of block B in the illustrated system.



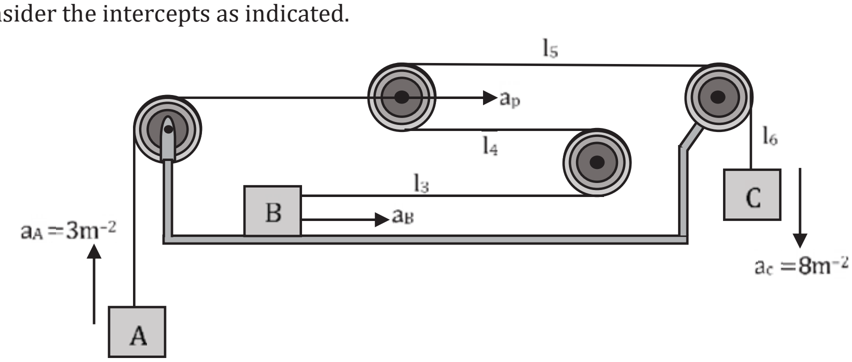
Sol. In the provided system, there are three mobile objects and one movable pulley. Assume the acceleration of block B as 'a' in the right direction. Segment each string into intercepts and formulate the constraint equations for them. Three equations are required for the three objects. Consider the acceleration as a variable for both the pulley and the blocks, as illustrated.



The constraint on acceleration for the initial string is expressed as

$$\ddot{l}_1 + \ddot{l}_2 = 0$$
$$-a_A + a_p = 0$$
$$a_A = a_p \quad \text{... (1)}$$

In the case of the second string,
Consider the intercepts as indicated.



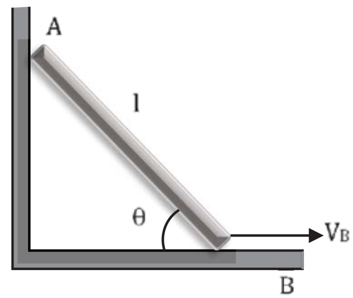
$$\ddot{l}_3 + \ddot{l}_4 + \ddot{l}_5 + \ddot{l}_6 = 0$$
$$-a_B - a_p - a_p + a_C = 0$$
$$a_B = a_C - 2a_p \quad \text{... (2)}$$

From (i) and (ii) $a_B = a_C - 2a_A$

Substituting the provided values of acceleration,

$$a_B = 8 - 2(3)$$
$$a_B = 2ms^{-2}$$

Ex. Determine the velocity of end A at the moment when the rod forms an angle θ with the horizontal.



Sol. As end B moves towards the right, end A will move downwards, and both ends will consistently stay in contact with the surface. Adopt the specified velocity variables and coordinate axes. Assume the coordinates of point B as (x,0) and the coordinates of point A as (0, y).In triangle ,

$$x^2 + y^2 = l^2 \text{ (Constant)}$$

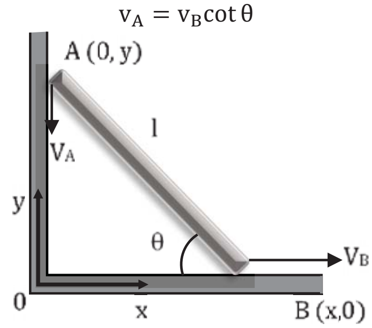
$$\frac{d(x^2)}{dt} + \frac{d(y^2)}{dt} = 0$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$-\frac{dy}{dt} = \left(\frac{x}{y}\right)\frac{dx}{dt} \qquad \text{... (1)}$$

$$\frac{dx}{dt} = v_B, \frac{dy}{dt} = -v_A \text{ and } \frac{x}{y} = \cot\theta$$

Putting in (1) $\quad v_A = v_B \cot\theta$

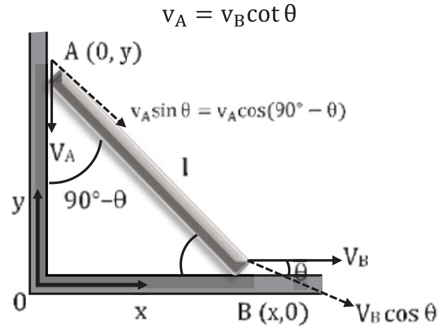


Alternative way:

The velocity components of the rod's ends along its length are depicted in the figure. By setting the components of the rod's velocity along its length equal to each other,

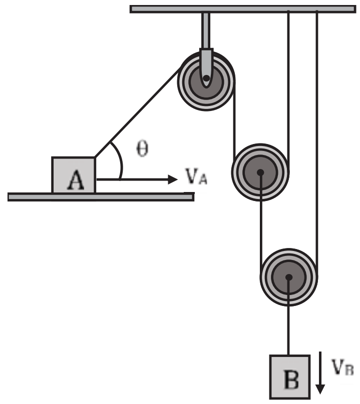
$$v_A \sin\theta = v_B \cos\theta$$

$$v_A = v_B \cot\theta$$

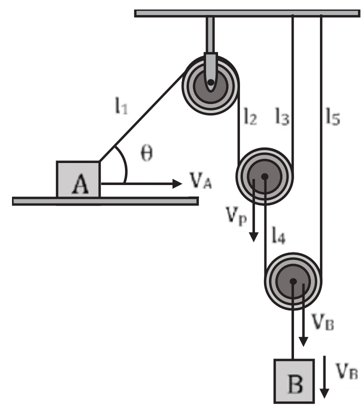


When the separation between two points is constant, the velocity components along the line connecting them must be equal.

Ex. Determine the connection between the velocities of blocks A and B.



Sol. In the provided system, there are two movable pulleys and two mobile blocks. Adopt the depicted velocities and variables as indicated in the figure.



$$\dot{l}_1 + \dot{l}_2 + \dot{l}_3 = 0$$

$$(-v_A\cos\theta) + v_p + v_p = 0$$

$$v_A\cos\theta = 2v_p \quad \ldots (1)$$

Likewise, for the second string

$$\dot{l}_4 + \dot{l}_5 = 0$$

$$+(v_B - v_p) + v_B = 0$$

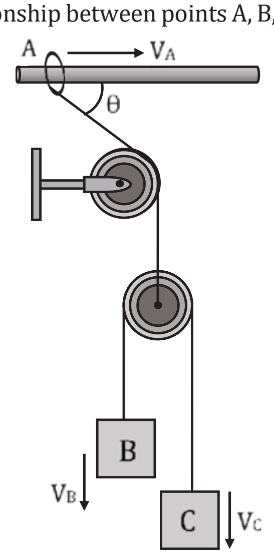
$$v_p = 2v_B \quad \ldots (2)$$

From (1) and (2)

$$v_A\cos\theta = 2(2v_B)$$

$$v_A\cos\theta = 4v_B$$

Ex. Determine the constraint relationship between points A, B, and C as a ring slides to the right on a rod.



Sol. As the ring slides, the movable pulley descends. Subsequently, based on the velocity components of the endpoints along the string's length,

$$v_A\cos\theta = v_P$$

For the movable pulley, applying the formula derived for pulley-block interaction,

$$\vec{v}_P = \frac{\vec{v}_1 + \vec{v}_2}{2}$$

Since both the pulley and the two blocks are descending, substitute the values of the velocity variables for the pulley, block C, and block B.

$$v_A \cos\theta = \frac{v_B + v_C}{2}$$