MAXIMA AND MINIMA, SOME RULES OF DIFFERENTIATION

Critical Points

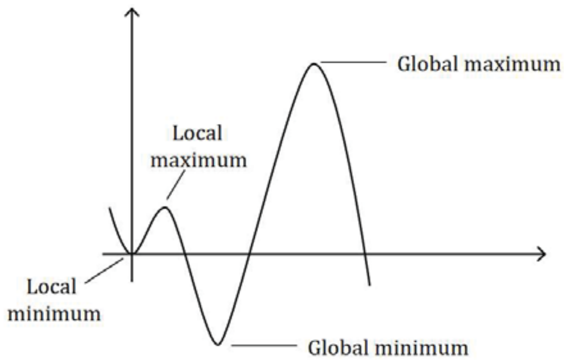
In calculus, the concept of a critical point holds significant importance, particularly in its widespread application to solving optimization problems.

At a critical point on the graph of a function, one can observe either a horizontal or vertical tangent. Drawing upon this observation, we can deduce additional insights about critical points.

A critical point of a function, represented as $y = f(x)$, corresponds to a juncture $(C, f(C))$ on the graph of the function $f(x)$. At this juncture, the first derivative of the function is either zero or undefined. Here, C denotes the x-coordinate of the function's point, while $f(C)$ indicates the value of the function at the critical point.

Maxima and Minima

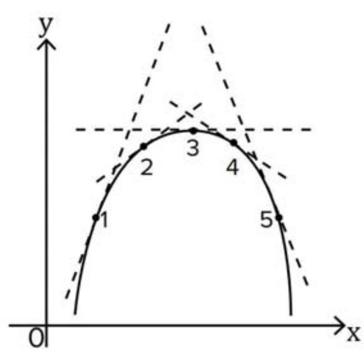
An extremum, alternatively referred to as a maximum or minimum, represents an extreme value of a function. Consider a function $y = f(x)$ defined over a specified domain of x. Depending on the interval of x where the function achieves an extremum, it can be categorized as either a 'local' or 'global' extremum. To gain a clearer understanding, let's delve deeper into the concept, particularly focusing on the case of maxima.



Conditions for maxima

a. $\frac{dy}{dx} = 0$

b. $\frac{d^2y}{dx^2} < 0$



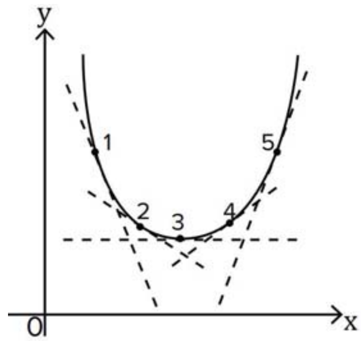
Look at the slope at points 1, 2, and 3. It is decreasing and becomes zero at 3.

For maxima, as x increases, the slope decreases.

Conditions for minima

a. $\frac{dy}{dx} = 0$

b. $\frac{d^2y}{dx^2} > 0$



Look at the slope at points 1, 2, and 3. It is increasing and becomes zero at 3.
For minima, as x increases, the slope increases.

Ex. Find the local maxima and minima for the function $y = x^3 - 3^x + 2$.

Sol. $\frac{dy}{dx} = 3x^2 - 3$

For critical points, we equate $\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 3 = 0 \Rightarrow (x+1)(x-1) = 0$

$x = 1$ and $x = -1$ are the critical points.

To find the maxima or minima, we calculate $\frac{d^2y}{dx}$.

$$\frac{d^2y}{dx} = 6x$$

For $x = 1, \frac{d^2y}{dx^2} = 6(1) = 6 \Rightarrow$ Positive. Hence, there is a minima at $x = 1$.

For $x = -1, \frac{d^2y}{dx^2} = 6(-1) = -6 \Rightarrow$ Negative. Hence, there is a maxima at $x = -1$.

Product Rule

$$\frac{d(f(x)\cdot g(x))}{dx} = \frac{df(x)}{dx}g(x) + \frac{dg(x)}{dx}f(x)$$

Ex. (a) $y = x.\,\text{Sin}\,x$ (b) $y = x^3.\,\text{Sin}\,x$

Sol. (a) $y = x.\,\text{Sin}\,x$

$$\frac{d}{dx}(y) = \frac{d}{dx}(x)\cdot \sin\ x + x\cdot\frac{d}{dx}(\sin\ x)$$
$$\frac{d}{dx}(y) = \sin\ x + x\cdot\cos\ x$$

(b) $y = x^3.\,\text{Sin}\,x$

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^3)\cdot \sin\ x + x^3\cdot\frac{d}{dx}(\sin\ x)$$
$$\frac{d}{dx}(y) = 3x^2\cdot\sin\ x + x^3\cdot\cos\ x$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{df(x)}{dx}g(x) - \frac{dg(x)}{dx}f(x)}{(g(x))^2}$$

Ex. (a) $y = \frac{\sin x}{x}$ (b) $y = \frac{x^2+1}{x^3}$

Sol. (a) $y = \frac{\sin x}{x}$

$$\frac{dy}{dx} = \frac{x\cdot\frac{d}{dx}(\sin\ x) - \sin\ x\cdot\frac{d}{dx}(x)}{x^2}$$
$$\frac{dy}{dx} = \frac{x\cdot\cos\ x - \sin\ x}{x^2}$$

(b) $y = \frac{x^2+1}{x^3}$

$$\frac{dy}{dx} = \frac{x^3 \cdot \frac{d}{dx}(x^2+1) - (x^2+1) \cdot \frac{d}{dx}(x^3)}{x^6}$$

$$\frac{dy}{dx} = \frac{2x^4 - (x^2+1) \cdot 3x^2}{x^2}$$

Composite Function and Chain Rule.

Chain Rule

If $f = f(g); g = g(x)$

$f'(x) = f'(g(x))g'(x)$

$g'(x)$ → Differentiate Inner function

$f'(g(x))$ → Differentiate Outer function

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

Composite function

$$f(x) = \sin(x)$$
$$g(x) = x^2$$
$$f[g(x)] = \sin(x^2)$$