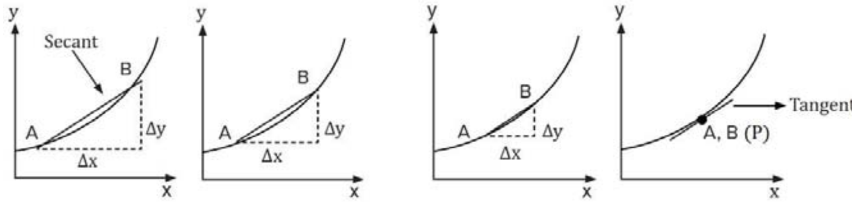
DIFFERENTIATION AND ITS APPLICATION

Introduction to differentiation (Tangent, Secant)

Tangent - The limiting case of secant



Tangent

- It is a line that touches the curve at one point.
- It is a limiting case of secant that intersects the curve at two infinitesimally close points.
- The derivative of a function at a point gives the slope of tangent at that point.
- Slope of a tangent at any point of a curve can also be given by, tan θ

  Where,

  θ = inclination of tangent with x-axis

  For tangent at P,

$$\text{Slope} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$
$$= \frac{dy}{dx} = y' = f'(x)$$

Differentiation

It involves examining how one amount changes concerning another. We can determine how fast a function changes by using a mathematical technique called differentiation.

It is denoted by the notation: $\frac{dy}{dx}$, where $y = f(x)$ is a function.

Derivative of function y is also written as y'

Example: $(\sin\ x)' =$ Differentiation of $\sin\ x = \frac{d(\sin\ x)}{dx}$

If we differentiate the derivative again, it will be known as a double derivative. It is expressed as:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Differentiation of polynomials

To calculate the derivative of a polynomial function, it's essential to become well-acquainted with the fundamental derivative formulas and rules outlined below. These formulas and rules are instrumental in computing the derivative of both straightforward and intricate polynomial functions.

$$\frac{d}{dx}(c) = 0$$
$$\frac{d}{dx}(x) = 1$$
$$\frac{d}{dx}(x^n) = nx^{n-1}$$
$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$
$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$
$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{d}{dx}}{v^2}$$

Certain functions incorporate trigonometric functions alongside variables. In such instances, it becomes necessary to determine the derivative of these trigonometric functions in conjunction with other terms. Thus, it is imperative to possess an understanding of how to compute the derivatives of these functions. Let's explore the process of finding the derivative of a trigonometric function, along with the associated formulas.

Differentiation of trigonometric functions

(1) $\frac{d}{dx}(\sin x) = \cos x;$

(4) $\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$

(2) $\frac{d}{dx}(\cos x) = -\sin x;$

(5) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(3) $\frac{d}{dx}(\tan x) = \sec^2 x;$

(6) $\frac{d}{dx}(\mathrm{cosec}\ x) = -\mathrm{cosec}\ x \cot x$

Differentiate $\sin \theta$ W.r.t x.

Here, the x variable is not present in the given function, thus given function becomes a constant, so,

$\frac{dy}{dx} = \frac{d(\sin \theta)}{dx} = 0(\because \theta \text{ is independent of x})$

If we were asked to differentiate w.r.t, then

$$\frac{dy}{d\theta} = \frac{d(\sin \theta)}{d\theta} = \cos \theta$$

Differentiation of exponential and logarithmic functions

These concepts fall within the domain of differential calculus, a branch of mathematics concerned with predicting the behavior or rate of change of variables or quantities. This predictive capacity aids in determining the rate at which a given input value changes. This predictive process is known as differentiation. Additionally, integration, which is the inverse operation of differentiation, plays a pivotal role here. Its utility extends across various fields, where the analysis of rate changes is paramount.

Derivatives, a product of differentiation, serve as indispensable tools for locating minima and maxima within functions, thereby facilitating the formulation and solution of differential equations. These equations encapsulate relationships between rates of change and the variables involved.

Differential analysis permeates numerous disciplines, including complex analysis, measure theory, functional analysis, differential geometry, and abstract algebra. In each of these fields, the application of differential analysis enables a deeper understanding of behavioral patterns and aids in the prediction of outcomes across diverse ranges of inputs.

Differentiation of exponential

An exponential function is a mathematical expression denoted as y = f(x) = bx.

where "x" represents a variable and "b" is a constant known as the base of the function, with the condition that $b > 1$. The prevailing choice for the base of an exponential function is often the transcendental number e, with a value of approximately 2.71828.

By employing the base "e," the exponential function can be expressed as y = ex, termed the natural exponential function. Conversely, an exponential function with a base of 10 is referred to as the common exponential function.

Logarithmic functions

Should the inverse of the exponential function exist, we can represent the logarithmic function as follows:

Let's consider a real number, $b > 1$, such that the logarithm of a to the base b is x if $b^x = a$. This relationship can be expressed as $\log_b a$.

Hence, $\log_b a = x$ if $b^x = a$.

In essence, by selecting a base $b > 1$, we can perceive the logarithm as a function mapping positive real numbers to all real numbers. This function is known as the logarithmic function and is defined as $\log_b$: $R^+ \to R$,

where x is mapped to $\log_b x = y$ if $b^y = x$.

If the base $b = 10$, it's termed a common logarithm; whereas if $b = e$, it's referred to as the natural logarithm. Typically, the natural logarithm is denoted by ln.

Properties of derivatives, second derivative

Properties of derivatives

Derivative of constant times a function

$$\frac{d}{dx}(af(x)) = a\frac{df(x)}{dx}$$

Derivative of sum or difference of two functions

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}$$

Important Formulas

$$\frac{d}{dx}(e^x) = e^x \frac{d}{dx}(\log_e x) = \frac{1}{x}\frac{d}{dx}(a^x) = a^x \log_e a$$

second derivative

Second-order derivatives play a crucial role in providing insights into the curvature and shape of a graph corresponding to a given function. They aid in categorizing functions based on their concavity, a measure of the curvature of the graph.

The concavity of a graph function can be categorized into two distinct types:

(i) **Concave Up:**

This classification refers to a graph that exhibits an upward curvature, resembling the shape of a bowl opening upwards.

(ii) **Concave Down:**

Conversely, a graph classified as concave down demonstrates a downward curvature, akin to the shape of a bowl opening downwards.

By analyzing the second-order derivatives of a function, one can discern whether the graph tends to curve upwards or downwards, thereby facilitating a comprehensive understanding of its overall shape and behavior.

Importance of Second Derivative

The significance of the second derivative becomes apparent when we delve into its role and implications.

While the first derivative provides insights into whether a given function is experiencing an increase or decrease, the second derivative offers additional depth by revealing whether the rate of change indicated by the first derivative is itself increasing or decreasing.

The application of the second derivative test extends to the identification of local maxima and minima in functions with one variable, as well as in those with multiple variables, albeit under specific conditions. Furthermore, this test proves instrumental in pinpointing inflection points or saddle points within a function.

Moreover, the second derivative test aids in determining the maximum or minimum values of a given function, thereby enriching our understanding of its critical points. Additionally, it allows us to visually discern the nature of the curve—whether it is concave or convex—through graphical analysis.

By employing the second derivative test, we gain insights into whether the given curve exhibits a concave-up or concave-down shape, thus enhancing our comprehension of its overall behavior and characteristics.

Applications

Derivative formulas hold extensive applicability across a myriad of disciplines, transcending the boundaries of mathematics alone. Industries such as science, engineering, computer science, and physics heavily lean on derivatives to confront dynamic challenges and tackle intricate problems.

Within the domain of mathematics, derivative formulas find predominant usage in various capacities:

Rate of Change of Quantity: Derivatives serve as indispensable tools for quantifying and comprehending the pace at which a quantity evolves over time or in reaction to different variables.

Tangent and Normal to a Curve: Derivatives play a pivotal role in determining the slope of a curve at a specific point, facilitating the computation of tangent and normal lines. These lines offer invaluable insights into the behavior and characteristics of the curve.

Newton's Laws: In Newtonian physics, derivatives are pivotal for expressing the interplay between force, acceleration, and mass as delineated by Newton's second law, $F = ma$.

Increasing and Decreasing Functions: Derivatives are instrumental in pinpointing intervals where a function experiences growth or decline, providing illumination on its overarching patterns and tendencies.

Minimum and Maximum Values: Through the analysis of critical points where the derivative is either zero or undefined, derivatives aid in the detection of a function's minimum and maximum values.

Linear Approximation: Derivatives facilitate the construction of linear approximations or tangent lines in proximity to a designated point on a curve. These approximations assist in estimating function values and offer invaluable insights in diverse contexts.

In essence, derivative formulas serve as indispensable instruments across a spectrum of fields, furnishing potent analytical capabilities and fostering deeper understandings of the behaviors exhibited by functions, systems, and phenomena.