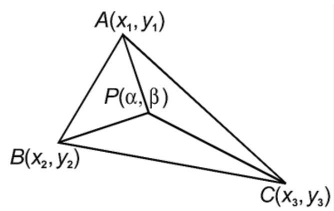
POSITION OF A POINT WHICH LIES INSIDE A TRIANGLE

1[st] method :

Let P (α, β) be a given point and the equation of the side BC, CA and AB of a given triangle ABC with vertices $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be



$$a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0 \text{ and } a_2x + b_2y + c_3 = 0$$

Respectively. Let P lie inside the triangle, then the points P and A are on the same side of BC, P and B are on the same side of AC, P and C are on the same side of AB and hence

$$(a_1\alpha + b_1\beta + c_1)(a_1x_1 + b_1y_1 + c_1) > 0 \quad \text{... (1)}$$
$$(a_2\alpha + b_2\beta + c_2)(a_2x_2 + b_2y_2 + c_2) > 0 \quad \text{... (2)}$$
$$(a_3\alpha + b_3\beta + c_3)(a_3x_3 + b_3y_3 + c_3) > 0 \quad \text{... (3)}$$

The required values of P (α, β) must be intersection of these inequalities (1), (2) and (3).

2[st] method :

Let us first draw the exact diagram of the problem. If the point P (α, β) move on the straight line $y = ax + b$ for all a, then $P = (a, ax + b)$ and the portion DE of the line $y = ax + b$ (Excluding D and E) lies within the triangle. Now line $y = ax + b$ cuts any two sides out of three sides, then we find coordinates of D and E.

$$D \equiv (x', y') \text{ and } E \equiv (x'', y''). \text{ Then } x' < \alpha < x'' \text{ and } y' < a\alpha + b < y''.$$

Ex. Find the possible values of a for which the point (a, a^2) lies inside the triangle formed by the straight line 2x + 3y-1=0, x+2y-3=0 and 5x - 6y - 1 = 0.

Sol **1[st] method**

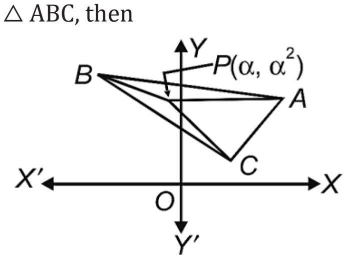
The equations of sides of a triangle ABC are given by

$$2x + 3y - 1 = 0, x + 2y - 3 = 0 \text{ and } 5x - 6y - 1 = 0.$$

The coordinates of the point of intersection (vertices) taken two by two are $A(\frac{5}{4}, \frac{7}{8})$

$$B(-7,5) \text{ and } C(\frac{1}{3}, \frac{1}{9})$$

Since $P(\alpha, \alpha^2)$ lies inside the $\triangle$ ABC, then



1. A and P must lie on the same side of BC

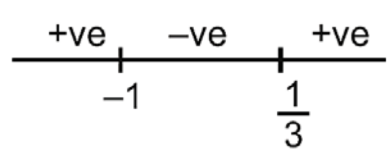
2. P and B must lie on the same side of CA and C & P must lie on the same side of AB, hence

$(\frac{5}{2} + \frac{21}{8} - 1)(2\alpha + 3\alpha^2 - 1) > 0$

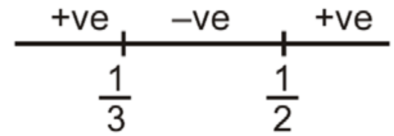
$3\alpha^2 + 2\alpha - 1 > 0$

$(\alpha + 1)(\alpha - \frac{1}{3}) > 0$

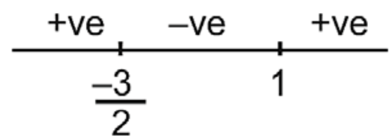
$\alpha \in (-\infty, -1) \cup (\frac{1}{3}, \infty)$ ... (1)



and $(-35-30-1)(5\alpha-6\alpha^2-1)>0$
$5\alpha-6\alpha^2-1<0$
$6\alpha^2-5\alpha+1>0$
$(\alpha-\frac{1}{2})(\alpha-\frac{1}{3})>0$
$\alpha\in(-\infty,\frac{1}{3})\cup(\frac{1}{2},\infty)$ ... (2)



Also, $(\frac{1}{3}+\frac{2}{9}-3)(\alpha+2\alpha^2-3)>0$
$2\alpha^2+\alpha-3<0$
$(2\alpha+3)(\alpha-1)<0$
$\alpha\in(-\frac{3}{2},1)$ ... (3)



From (1), (2), and (3) we get,
$\alpha\in(-\frac{3}{2},-1)\cup(\frac{1}{2},1)$.

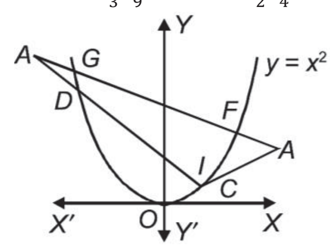
2^{st} method :

The point $P(\alpha,\alpha^2)$ move on the curve $y=x^2$ for all α.
The intersection of $y=x^2$ and $2x+3y-1=0$. i.e., $2x+3x^2-1=0$ is $x=-1,\frac{1}{3}$
The points of intersection are $D(-1,1)$ and $E(\frac{1}{3},\frac{1}{9})$.
Similarly the intersection of $y=x^2$ and $x+2y-3=0$ or $x+2x^2-3=0\Rightarrow x=1,x=\frac{-3}{2}$
Let the intersection points $F\equiv(1,1)$ and $G(-\frac{3}{2},\frac{9}{4})$ and intersection of $y=x^2$ and

$$5x-6y-1=0\Rightarrow 5x-6x^2-1=0$$
$$x=\frac{1}{3},\frac{1}{2}$$

Let intersection points be

$$H\equiv(\frac{1}{3},\frac{1}{9})\text{ and }I=(\frac{1}{2},\frac{1}{4})$$



Thus the points on the curve $y=x^2$ whose coordinates lies between $-\frac{3}{2}$ and -1 and $\frac{1}{2}$and 1lies within the triangle ABC.

Consequently, $-\frac{3}{2}<\alpha<-1$ and $\frac{1}{2}<\alpha<1$

$$\alpha\in(\frac{-3}{2},-1)\cup(\frac{1}{2},1)$$