PAIR OF STRAIGHT LINES

Let us consider the equations $xy = 0, x^2 + 6y^2 - 5xy = 0$ and $(x + y - 1)(2x + 3y + 5) = 0$

From the first equation, we have $x = 0$ or $v = 0$ and we know that $x = 0$ represents y-axis whereas $y = 0$ represents x-axis. Thus $xy = 0$ represents two straight lines. Similarly the equations $x^2 + 6y^2 - 5xy = 0$ and $(x + y - 1)(2x + 3y + 5) = 0$ represent straight lines $x - 3y = 0$ $x - 2y = 0$ and $x + y - 1 = 0$ $2x + 3y + 5 = 0$ respectively. Thus it can be concluded that equation representing two straight lines is of second degree equation (homogeneous and non-homogeneous) in two independent variables x and y. Let us consider homogeneous equation of second degree in x & y given by.

$$ax^2 + 2hxy + by^2 = 0, a \neq 0$$
$$a^2x^2 + 2ax \cdot hy + h^2y^2 = h^2y^2 - aby^2$$
$$(ax + hy)^2 = (h^2 - ab)y^2$$
$$ax + hy = \pm\sqrt{h^2 - ab} \cdot y$$
$$ax + (h - \sqrt{h^2 - ab})y = 0 \text{ and } ax + (h + \sqrt{h^2 - ab})y = 0$$
$$ax^2 + 2hxy + by^2 = 0 \text{ will represent two straight lines}$$
$$ax + (h - \sqrt{h^2 - ab})y = 0 \text{ and } ax + (h + \sqrt{h^2 - ab})y = 0$$

Passing through origin provided $h^2 - ab \geq 0$

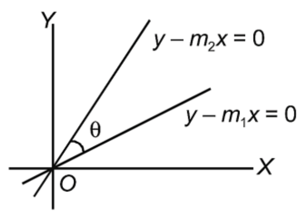
Let straight lines represented by $ax^2 + 2hxy + by^2 = 0$ be $y - m_1x = 0$ and $y - m_2x = 0$

Then

$$ax^2 + 2hxy + by^2 \equiv (y - m_1x)(y - m_2x) = 0$$
$$\equiv y^2 - (m_1 + m_2)xy + m_1m_2x^2 = 0$$

$m_1 + m_2 =$ Sum of gradients of the straight lines $= \frac{-2h}{b}$ and

$m_1m_2 =$ Product of gradients of the represented straight lines $= \frac{a}{b}$



Angel Between the lines represented by $ax^2 + 2hxy + by^2 = 0$

Let the straight line represented by $ax^2 + 2hxy + by^2 = 0$ be $y - m_1x = 0$ and $y - m_2x = 0$

$$m_1 + m_2 = \frac{-2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$

Let be θ the angel between the represented straight lines.

$$\theta = (\theta_2 - \theta_1)$$
$$\tan\theta = \tan(\theta_2 - \theta_1)$$
$$\frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1\tan\theta_2} = \frac{m_2 - m_1}{1 + m_1m_2}$$
$$\pm\frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2}$$
$$\pm\frac{2\sqrt{h^2 - ab}}{a + b}$$

Thus the acute angel between the straight lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by

$$\tan^{-1}\left(\left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|\right)$$

The two lines will be coincident if $\theta = 0 \Rightarrow h^2 - ab = 0$ and perpendicular iff $\theta = \frac{\pi}{2} \Rightarrow a + b = 0$

Sum of coefficients of x^2 and $y^2 = 0$

Ex. Find the straight lines represented by the following equations and also determine the angle between them.

(a) $x^2 + 2xy\sec\theta + y^2 = 0$ **(b)** $24x^3 - 14x^2y - xy^2 + y^3 = 0$

Sol. **(a)** We have

$$x^2 + 2xy\sec\theta + y^2 = 0$$
$$x^2 + 2x \cdot y\sec\theta + y^2\sec^2\theta = y^2\sec^2\theta - y^2$$
$$(x + y\sec\theta)^2 = y^2(\sec^2\theta - 1) = y^2\tan^2\theta$$
$$x + y\sec\theta = \pm y\tan\theta$$
$$x + (\frac{1}{\cos\theta} \mp \frac{\sin\theta}{\cos\theta})y = 0$$
$$x\cos\theta + (1 - \sin\theta)y = 0$$
$$x\cos\theta + (1 + \sin\theta)y = 0$$

Thus, the straight lines represented by the given equation are given as

$$x\cos\theta + (1 - \sin\theta)y = 0 \text{ and } x\cos\theta + (1 + \sin\theta)y = 0$$

The angel between them

$$\tan^{-1}(\frac{2\sqrt{\sec^2\theta - 1}}{1+1}) = \tan^{-1}(\tan\theta) = \theta$$

(b) We have

$$y^3 - xy^2 - 14x^2y + 24x^3 = 0$$

Dividing the equation by x^3, we get

$$\frac{y^3}{x^3} - (\frac{y}{x})^2 - 14\frac{y}{x} + 24 = 0$$

Factorizing by remainder theorem, we get

$$(\frac{y}{x} - 2)(\frac{y}{x} - 3)(\frac{y}{x} + 4) = 0 \Rightarrow (y - 2x)(y - 3x)(y + 4x) = 0$$

Hence the line represented by the given equation are $y - 2x = 0, y - 3x = 0$ and

$$y + 4x = 0$$

Let θ_1 be the angle between $y - 2x = 0$ and $y - 3x = 0$, then

$$\tan\theta_1 = \frac{3-2}{1+3\cdot2} = \frac{1}{7} \Rightarrow \theta_1 = \tan^{-1}(\frac{1}{7})$$

Let θ_2 be the angle between $y - 2x = 0$ and $y + 4x = 0$, then

$$\tan\theta_2 = \frac{2-(-4)}{1+2(-4)} = \frac{-6}{7} \Rightarrow \theta_2 = \tan^{-1}(\frac{-6}{7})$$

Let θ_3 be the angle between $y - 3x = 0$ and $y + 4x = 0$

$$\tan\theta_3 = \frac{3-(-4)}{1+3(-4)} = \frac{-7}{11}$$
$$\theta_3 = \tan^{-1}(\frac{-7}{11})$$

Joint Equation of Pair of Straight Lines Joining Origin and the Points of Intersection of a Curve and a Line

The equation of the curve be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

And the straight line be $\ell x + my + n = 0$.

$$\frac{\ell x + my}{-n} = 1$$

Consider points P and Q as the intersections between the provided straight line and the curve. To obtain the equation of the pair of lines connecting the origin to these points of intersection, homogenize the equation of the curve using the equation of the line. Consequently, the resulting equation represents the sought-after pair of lines.

$$ax^2 + 2hxy + by^2 + 2gx(\frac{\ell x + my}{-n}) + 2fy(\frac{\ell x + my}{-n}) + (\frac{\ell x + my}{-n})^2 c = 0$$

Ex. **1.** Prove that the pair of lines Joining the origin to the intersection of the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by the line $l\, x + my + n = 0$ are coincident if $a^2 l^2 + b^2m^2 = n^2$

2. Show that the straight line joining the origin to the other two points of intersection of the curves whose equations are $ax^2 + 2hxy + by^2 + 2fy = 0$ and $a'x^2 + 2h'xy + b'y^2 + 2f'y = 0$ will be at right angled if $f(a' + b') = f'(a + b)$

Sol. **1.** The equation of the pair of straight lines joining the origin to the intersection of the curve. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ by the straight line $(x+my+n=0$ is given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=(\frac{\ell x+my}{-n})^2$

$$(\frac{1}{a^2}-\frac{\ell^2}{n^2})x^2-\frac{2\ell m}{n^2}xy+y^2(\frac{1}{b^2}-\frac{m^2}{n^2})=0$$

Which will represent coincident lines iff

$$\frac{\ell^2m^2}{n^4}-(\frac{1}{a^2}-\frac{\ell^2}{n^2})(\frac{1}{b^2}-\frac{m^2}{n^2})=0$$

$$0=\frac{1}{a^2b^2}-\frac{\ell}{n^2b^2}-\frac{m^2}{a^2n^2}$$

$$a^2\ell^2+b^2m^2=n^2$$

2. The pair of lines joining the origin to the points of intersection of $ax^2+2hxy+by^2+2fy=0$ and $ax^2+2h'xy+b'y^2+2f'y=0$ can be obtained by making them homogeneous.

Multiplying the first equation by f' and second by f and then subtracting we get,

$$(af'-a'f)x^2+2(hf'-2h'f)xy+(bf'-b'f)y^2=0$$

Which will represent two perpendicular straight lines iff

Coefficient of x^2+ coefficient of $y^2=0$

$$af'-a'f+bf'-b'f=0$$

$$(a+b)f'=(a'+b')f$$