IMAGE REFLECTION, FOOT OF PERPENDICULAR, PERPENDICULAR DISTANCE OF A POINT WITH RESPECT TO A LINE

The perpendicular distance from a point to a line can be determined when the equation of the line and the coordinates of the point are provided.

Case I.

First, let's establish the formula for this scenario when the equation of the line is presented in normal form.

Consider the equation of the line, denoted as 'l,' in normal form as:

$$x\cos\alpha + y\sin\alpha = p$$

When α represents the angle formed by the perpendicular line from the origin to the line with the positive direction of the x-axis, and 'p' is the length of this perpendicular, let's consider a point $P(x_1, y_1)$ that doesn't lie on the line 'l.' Suppose the perpendicular line drawn from point P to line 'l' is PM, with PM measuring a distance 'd.' We assume that point P is situated on the opposite side of line 'l' from the origin O. Next, draw a line 'l'' parallel to 'l' passing through point P. Let ON be the perpendicular drawn from the origin to line 'l,' and it intersects line 'l'' at point R.

Obviously $\quad ON = p$ and $\angle XON = \alpha$

Also, as observed in the figure, we can see that

$$OR = ON + NR = p + MP$$

Hence, the length of the perpendicular from the origin to line 'l'' is

$$OR = p + d$$

The angle formed by the perpendicular OR with the positive x-axis is α. Therefore, the equation of line 'l'' in normal form can be expressed as

$$x\cos\alpha + y\sin\alpha = p + d$$

As the line 'l'' passes through point P, the coordinates (x_1, y_1) of point P must satisfy the equation of line 'l'', resulting in

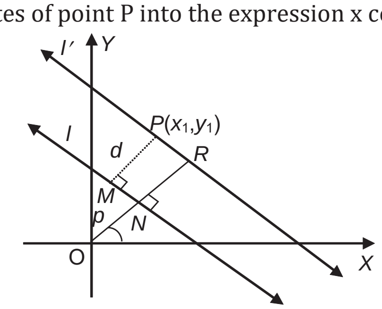
$$x_1\cos\alpha + y_1\sin\alpha = p + d$$

$$d = x_1\cos\alpha + y_1\sin\alpha - p$$

The length of a segment is inherently non-negative. Thus, we consider the absolute value of the right-hand side (RHS)

$$d = |x_1\cos\alpha + y_1\sin\alpha - p|$$

Therefore, the length of the perpendicular is the absolute value of the outcome obtained when substituting the coordinates of point P into the expression x cos α + y sin α - p.



Case II.

Assuming the equation of the line is in the form of

$$Ax + By + C = 0$$

Simplifying the general equation to the normal form is achieved by:

$$\pm\frac{A}{\sqrt{A^2+B^2}}x \pm \frac{B}{\sqrt{A^2+B^2}}y = \pm\frac{C}{\sqrt{A^2+B^2}}$$

Where the sign is selected as either + or - in order to ensure that the right-hand side (RHS) is positive.

(a) When C < 0

In this scenario, the normal form of the line's equation is given by:

$$\frac{A}{\sqrt{A^2+B^2}}x + \frac{B}{\sqrt{A^2+B^2}}y = -\frac{C}{\sqrt{A^2+B^2}}$$

Now, based on the outcome of case (I), the length of the perpendicular segment extending from point P(x_1, y_1) to line (II) is

$$d = \left|\frac{A}{\sqrt{A^2+B^2}}x_1 + \frac{B}{\sqrt{A^2+B^2}}y_1 + \frac{C}{\sqrt{A^2+B^2}}\right|$$

$$d = \left|\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}\right|$$

(b) When C > 0

In this situation, the normal form of equation (II) takes the following shape:

$$-\frac{A}{\sqrt{A^2+B^2}}x - \frac{B}{\sqrt{A^2+B^2}}y = \frac{C}{\sqrt{A^2+B^2}}$$

Again, according to the outcome in case (I), the distance 'd' can be expressed as:

$$d = \left|\frac{-Ax_1-By_1-C}{\sqrt{A^2+B^2}}\right|$$

$$d = \left|\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}\right|$$

The perpendicular distance of point (x_1, y_1) from the line represented by $ax + by + c = 0$ can be calculated as:

$$d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

Therefore, the perpendicular distance from the origin is $\frac{|c|}{\sqrt{a^2+b^2}}$.

Ex. The equation of the base of an equilateral triangle is $x + y = 2$, and one of its vertices is located at (2, -1). Determine the length of one side of the triangle.

Sol. Equation of side BC is

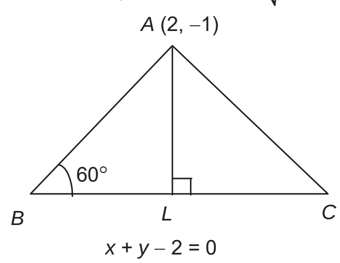
$$x + y - 2 = 0 \qquad \dots (1)$$

$$A \equiv (2, -1)$$

$$AL = \left|\frac{2-1-2}{\sqrt{1^2+1^2}}\right| = \frac{1}{\sqrt{2}}$$

From ΔABL, $\sin 60^\circ = \frac{AL}{AB}$

$$\frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}AB} \text{ or } AB = \sqrt{\frac{2}{3}}$$



Distance between parallel lines

The formula for calculating the distance between two lines represented by $ax + by + c = 0$ and $ax + by + c' = 0$ is as follows: $d = \frac{|c-c'|}{\sqrt{a^2+b^2}}$

Ensure that the coefficients of x and y are identical in both lines before using the formula.

Ex. Determine the distance between the lines represented by $5x + 12y + 40 = 0$ and $10x + 24y - 25 = 0$.

Sol. In this case, the coefficients of x and y differ between the two equations. Therefore, we can express them as

$$5x + 12y + 40 = 0$$

$$5x + 12y - \frac{25}{2} = 0$$

Now, distance between them $= \frac{|40-(-\frac{25}{2})|}{\sqrt{(5)^2+(12)^2}}$

$$= \frac{105}{2(13)} = \frac{105}{26}$$