Chapter 9

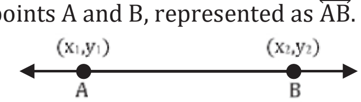
Straight Lines

INTRODUCTION

A straight line is an infinite one-dimensional shape that possesses no width. It is formed by an unending series of points connected on both sides of a central point. A straight line exhibits no curvature and can take the form of a horizontal, vertical, or slanted orientation. When an angle is drawn between any two points along a straight line, it consistently measures 180°. In this brief tutorial, we will delve into the realm of straight lines, examining the various formats of their equations and learning how to address problems related to them.

A **straight line** is an unending, curve-free line of infinite length. It can be created by connecting two points, with the line extending indefinitely in both directions. This linear figure takes shape when we join two points, A (x_1, y_1) and B (x_2, y_2), with the shortest distance between them, and extend the line infinitely in both directions.

In the illustrated diagram below, you can see a straight line connecting points A and B, represented as $\overleftrightarrow{AB}$.



Although straight lines lack a fixed starting or ending point, we frequently encounter them in our everyday experiences, as seen in instances like railway tracks or highways.

Types of Straight Lines

Straight lines come in diverse forms, typically categorized by their orientation, which is determined by the angle they make with the x-axis or y-axis. Based on their orientation, straight lines can be classified into the following types:

- Horizontal lines
- Vertical lines
- Oblique or Slanted lines

We will now delve into each of these categories to gain a better understanding.

- Image Reflection, Foot of Perpendicular, Perpendicular Distance of a Point with Respect to a Line
- Analysis of Two Lines
- Equation of the Angle Bisectors
- Analysis of Three Lines
- Pair of Straight Lines
    - ➢ Pair of Straight Lines Passing through Origin
    - ➢ Angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$
    - ➢ Joint Equation of Pair of Straight Lines Joining Origin and the Points of Intersection of a Curve and a Line

Horizontal Lines

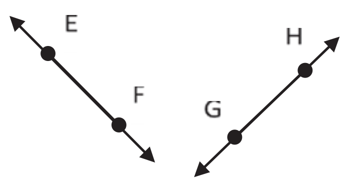
Horizontal lines are those drawn parallel to the x-axis or perpendicular to the y-axis. They create a 0° or 180° angle with the x-axis and a 90° or 270° angle with the y-axis. In the provided illustration, the line $\overleftrightarrow{AB}$ represents a horizontal line.

Vertical Lines

Vertical lines are those drawn parallel to the y-axis or perpendicular to the x-axis. They create a 90° or 270° angle with the x-axis and a 0° or 180° angle with the y-axis. In the provided illustration, the line $\overleftrightarrow{CD}$ represents a vertical line.

Oblique or Slanted Lines

Lines that are drawn at an inclined or slanting angle, forming angles other than 0°, 90°, 180°, 270°, or 360° with respect to the horizontal or vertical lines, are referred to as oblique or slanting lines. In the provided illustration, the lines $\overleftrightarrow{EF}$ and $\overleftrightarrow{GF}$ represent slanted lines.



Properties of a Straight Line

Below are the characteristics of straight lines:

- A straight line possesses an infinite length, and it is impossible to measure the distance between its two farthest points.
- It has zero area and zero volume, yet its length is infinite.
- A straight line is a one-dimensional shape.
- While an infinite number of lines can traverse through a single point, only one unique line passes through two distinct points.

1. Equation of straight line

The equation of a straight line is a relationship between x and y, and it holds true for the coordinates of every point situated on that line. In fact, any linear equation involving the variables x and y invariably corresponds to a straight line.

e.g. $3x + 4y = 5$, $-4x + 9y = 3$ etc.

The general expression for a straight line is represented as:

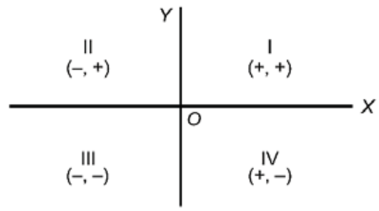
$$ax + by + c = 0.$$

2. Equation of straight line parallel to axes

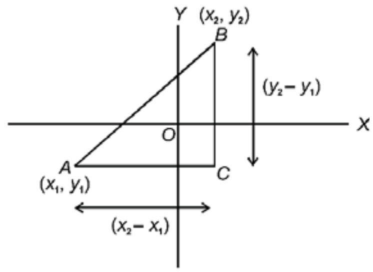
1. The equation for the **x-axis** can be expressed as $y = 0$.
   For a line parallel to the x-axis (or perpendicular to the y-axis) and situated at a distance 'a' from it, the equation can be given as $y = a$.
2. The equation for the **y-axis** is $x = 0$.
   For a line parallel to the y-axis (or perpendicular to the x-axis) at a distance 'a' from it, the equation becomes $x = a$.
   For instance, if we want to represent a line that is parallel to the x-axis and located 4 units in the negative direction, the equation is $y = -4$.

2. CARTESIAN COORDINATES

1. Analysis of Quadrant



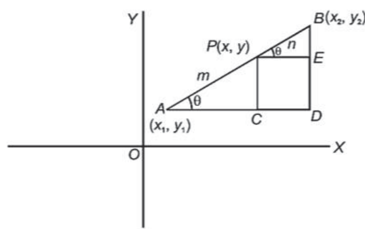
2. Distance Formula



$$AB = \sqrt{AC^2 + BC^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

3. Section formula

(a) Internal division: P(x, y) divides A (x_1, y_1) and B (x_2, y_2) in the ratio m: n,



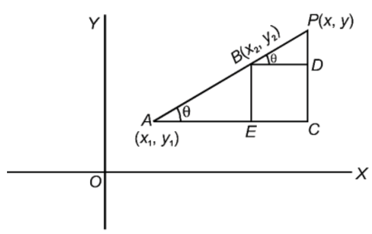
In similar triangle ACP and PEB (angle, angle, angle)

$$\frac{AP}{BP} = \frac{AC}{PE} \Rightarrow \frac{m}{n} = \frac{x - x_1}{x_2 - x} \Rightarrow x = \frac{mx_2 + nx_1}{m+n}$$

$$\text{Similarly } \frac{m}{n} = \frac{y - y_1}{y_2 - y} \Rightarrow y = \frac{my_2 + ny_1}{m+n}$$

Hence $$P \equiv \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right) \quad \ldots (1)$$

(b) External Division



Here $\frac{AP}{PB} = \frac{m}{n} \Rightarrow$ P divides AB externally

In similar triangle APC and BPD

$$\frac{AP}{PB} = \frac{AC}{BD} \Rightarrow \frac{m}{n} = \frac{x - x_1}{x - x_2} \Rightarrow x = \frac{mx_2 - nx_1}{m-n}, m \neq n$$

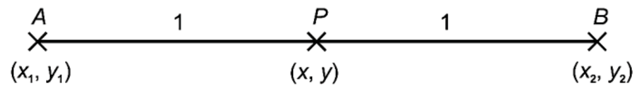
Similarly $$\frac{AP}{BP} = \frac{PC}{PD} \Rightarrow \frac{m}{n} = \frac{y - y_1}{y - y_2} \Rightarrow y = \frac{my_2 - ny_1}{m-n}, m \neq n$$

$$P \equiv \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right), m \neq n$$

(c) Middle Point

Put $m = n = 1$, in (1)

$$P \equiv \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

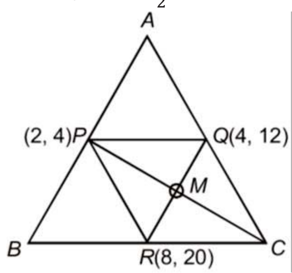


Ex. Determine the distance between the points "$(x_1, 0)$" and "$(x_2, 0)$".

Sol. Distance $=\sqrt{(x_1-x_2)^2+(0-0)^2}=|x_1-x_2|$ or $|x_2-x_1|$.

Ex. Given the midpoints of the sides of triangle ABC as (2, 4), (4, 12), and (8, 20), determine the vertices A, B, and C.

Sol. Since $PQ=\frac{1}{2}BC$ and $PQ \parallel BC$, similarly $QR=\frac{1}{2}AB, QR \parallel AB, PR=\frac{1}{2}AC$ and $PR \parallel AC$.



Hence PQCR is a parallelogram. Hence diagonal bisect. In symbolic form

$$M=\frac{P+C}{2}=\frac{Q+R}{2} \Rightarrow C=(Q+R-P)=(4+8-2,12+20-4)=(10,28)$$

Similarly, $A=2Q-C=(8-10,24-28)=(-2,-4)$

$B=2P-A=(4+2,8+4)=(6,12)$

4. Area Of Polygon

(a) If the vertices of a triangle are $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ then areaA, where

$$A=|\frac{1}{2}(x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2))|$$

$$=|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}|=|\frac{1}{2}[x_1(y_2-y_3)-y_1(x_2-x_3)+1(x_2y_3-x_3y_2)]|$$

$$=|\frac{1}{2}\left|\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{array}\right||=|\frac{1}{2}[(x_1y_2+x_2y_3+x_3y_1)-(x_2y_1+x_3y_2+x_1y_3)]|$$

(b) If $(x_1,y_1),(x_2,y_2),(x_3,y_3)\ldots\ldots\ldots(x_n,y_n)$ be the vertical of a polygon then its area $=|A|$ where

$$A=\frac{1}{2}\left|\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \\ x_n & y_n \\ x_1 & y_1 \end{array}\right|=\frac{1}{2}[(x_1y_2+x_2y_3+\cdots.+x_ny_1)-(x_1y_n+x_ny_{n-1}+\cdots+x_3y_2+x_2y_1)]b$$

> The point for the above formula must be in order.

5. Condition for collinearity

If $A(x_1,y_1),B(x_2,y_2),C(x_3,y_3)$ are collinear then

(a) Either $AB+BC=AC$

$AC+CB=AB$

$CA+AB=CB$ will be true

(b) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$

Ex. Find the values of h if the area of the triangle formed by the points (1, 1), (h, h), and (2, 3) is equal to 2.

Sol. $A = \frac{1}{2}\begin{vmatrix} h & h & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix}$

$$= \frac{1}{2}[h(1-3) - h(1-2) + 1(3-2)]$$

$$= \frac{1}{2}[-2h + h + 1] = \frac{1-h}{2}$$

$$\text{Area} = |A| = \left|\frac{1-h}{2}\right| = 2 \Rightarrow h - 1 = \pm 4$$

$$h = 5, -3$$

Ex. Determine the value of k such that the points (k, k), (2, 2), and (4, 4) lie on the same straight line.

Sol. For collinearity $\begin{vmatrix} k & k & 1 \\ 2 & 2 & 1 \\ 4 & 4 & 1 \end{vmatrix} = 0$

$$k(2-4) - k(2-4) + 1(8-8) = 0$$

Hence $k \in R.$

6. Different center of a triangle

(ABC having vertices $A \equiv (x_1, y_1), B \equiv (x_2, y_2), C \equiv (x_3, y_3)$)

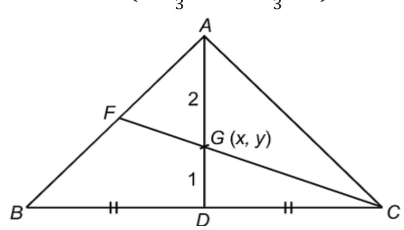
(a) Centroid (G): Centroid of a triangle is the point of intersection of medians.

$$D \equiv \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$$

$$\frac{AG}{GD} = \frac{2}{1} \text{ as centroid divides AD in the ratio } 2:1.$$

$$x = \frac{2\left(\frac{x_2+x_3}{2}\right)+x_1}{2+1}, y = \frac{2\left(\frac{y_2+y_3}{2}\right)+y_1}{2+1}$$

$$G \equiv \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$



(b) Incentre (I): Incentre is the point of intersection of bisectors of internal angles. With usual notations BC = a, AC = b, AB = c.

As we know that AD divides BC in the ratio c: b.

$$\frac{BD}{DC} = \frac{c}{b}$$

By section formula $D = \left(\frac{cx_3 + bx_2}{c+b}, \frac{cy_3 + by_2}{c+b}\right) = (x_4, y_4)$

$$\frac{BD}{DC} = \frac{c}{b} \Rightarrow \frac{BD+DC}{DC} = \frac{b+c}{b}$$

$$\frac{a}{DC} = \frac{b+c}{b} \Rightarrow DC = \frac{ab}{b+c}, \text{ similarly } BD = \frac{ca}{b+c}$$

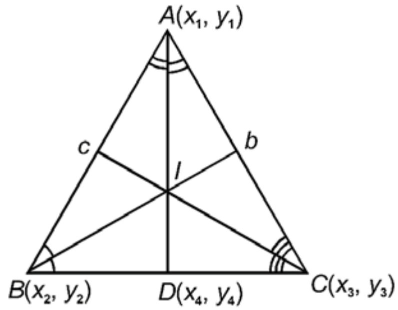
If we observe the triangle ABD, B is the bisector of B. Hence

$$\frac{AI}{ID} = \frac{c}{BD} = \frac{c}{\left(\frac{ca}{b+c}\right)} = \frac{b+c}{a}$$

If $I = (x, y)$, then $x = \frac{(b+c)x_4+ax_1}{b+c+a} = \frac{ax_1+bx_2+cx_3}{a+b+c}$

Similarly, $y = \frac{(b+c)y_4+ay_1}{a+b+c} = \frac{ay_1+by_2+cy_3}{a+b+c}$

$$I \equiv \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$



Ex. If the sides of triangle ABC are given by a = 3, b = 4, and c = 5, determine the ratio in which the in center divides the angle bisector BE.

Sol. Ratio $= \frac{c+a}{b} = \frac{3+5}{4} = \frac{2}{1}$

Ex. Find the in centre of triangle formed by A (1, 1), B (4, 1), C (1, 5). a = BC = 5, b = CA = 4, c = AB = 3.

Sol. a = BC = 5, b = CA = 4, c = AB = 3

$$(x_1, y_1) = (1,1), (x_2, y_2) = (4,1), (x_3, y_3) = (1,5)$$

$$I = \left(\frac{5\times1+4\times4+3\times1}{5+4+3}, \frac{5\times1+4\times1+3\times5}{5+4+3}\right) = (2,2)$$

7. Some Important Points about the Quadrilateral

(a) In a parallelogram, the diagonals are unequal, while opposite sides are equal.

(b) In a rectangle, the diagonals are equal, and opposite sides are also equal.

(c) In a rhombus, all sides are equal, but the diagonals are unequal.

(d) In a square, both sides and diagonals are equal.

(e) In any parallelogram, the diagonals bisect each other.