PRINCIPLES OF INCLUSION AND EXCLUSION

In the Venn's diagram **(i)**,

We get $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

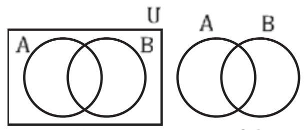
$$n(A' \cap B') = n(\cup) - n(A \cup B)$$

In the Venn's diagram **(ii)**,

We get

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

$$n(A' \cap B' \cap C') = n(\cup) - n(A \cup B \cup C)$$



(i) (ii)

In general, we have n $n(A_1 \cup A_2 \cup \ldots \ldots \ldots \cup A_n)$

Ex. Determine the number of permutations of the letters a, b, c, d, e, f, g taken all at once, ensuring that neither the 'beg' nor 'cad' pattern appears.

Sol. The total number of unrestricted permutations, denoted as $n(\cup)$, is equal to 7!

Let A be the set of all possible permutations in which the 'beg' pattern always appears:

$$n(A) = 5!$$

Let B be the set of all possible permutations in which the 'cad' pattern always appears:

$$n(B) = 5!$$

$n(A \cap B)$: Number of all possible permutations when both 'beg' and 'cad' patterns appear.

$n(A \cap B) = 3!$. Hence, the total number of permutations in which neither 'beg' nor 'cad' patterns appear is.

$$n(A' \cap B') = n(\cup) - n(A \cap B) = n(\cup) - n(A) - n(B) + n(A \cap B)$$

$$= 7! - 5! - 5! + 3!.$$

Arrangements

If ${}^nP_r$ denotes the number of permutations (arrangements) of n different things, taking r at a time, ${}^nP_r = n(n-1)(n-2)\ldots\ldots(n-r+1) = \frac{n!}{(n-r)!}$

De Arrangement

The number of ways to place n letters in n corresponding envelopes, such that no letter is placed in the correct envelope, is:

$$n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + \cdots \ldots + \frac{(-1)^n}{n!}\right]$$

Proof:

The set of distributions of n letters in envelopes, denoted by A_i, where each letter is placed in the corresponding envelope, is such that the i^{th} letter is placed in the correct envelope. The number of ways to achieve this is given by $n(A_i) = 1 \times (n^{-1})!$ [Since the remaining n^{-1} letters can be placed in n^{-1} envelopes in $(n^{-1})!$ ways].

Then, $n(A_i \cap A_j)$ represents the number of ways where letters i and j can be placed in their corresponding envelopes.

$$n(A_i \cap A_j) = 1 \times 1 \times (n-2)!$$

$$n(A_i \cap A_j \cap A_k) = 1 \times 1 \times 1 \times (n-3)!$$

Hence, the required number is

$$n(A_1 \cup A_2 \cup \ldots \ldots . \cup A_n) = n! - n(A_1 \cap A_2 \cap \ldots \ldots . \cap A_n)$$

$$n! - [\sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) + \cdots \ldots + (-1)^n \sum n(A_i \cap A_2 \ldots \cap \cap A_n)]$$

$$= n! - [{}^nC_1(n-1)! - {}^nC_2(n-2)! + {}^nC_3(n-3)! + \ldots\ldots. + (-1)^{n-1} \times {}^nC_n 1]$$

$$= n! - [\frac{n!}{1!(n-1)!}(n-1)! - \frac{n!}{2!(n-2)!}(n-2)! + \cdots \ldots + (-1)^{n-1}]$$

$$= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + \cdots \ldots \ldots + \frac{(-1)^n}{n!}\right]$$

Ex. A person is writing letters to six friends and addressing corresponding envelopes. Determine the number of ways the letters can be placed in the envelopes so that:

1. All the letters are in the wrong envelopes.
2. At least two of them are in the wrong envelopes.

Sol. 1. The number of ways is which all letters be placed in wrong envelopes

$$= 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$$

$$= 720\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right)$$

$$= 360 - 120 + 30 - 6 + 1 = 265.$$

2. The number of ways in which at least two of them in the wrong envelopes

$$= {}^6C_4 . 2!\left(1 - \frac{1}{1!} + \frac{1}{2!}\right) + {}^6C_3 . 3!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) + {}^6C_2 . 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

$$+ {}^6C_1 . 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) + {}^6C_0\, 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$$

$$= 15 + 40 + 135 + 264 + 265 = 719$$