PERMUTATION

A PERMUTATION refers to each arrangement in a specific order that can be formed by selecting some or all of the items at a time. In permutations, the order of the items is considered significant; changing the order results in a different permutation. Typically, permutation problems involve arrangements such as standing in a line, being seated in a row, digit problems, letter problems from a word, and so on.

${}^{n}P_r$ Denotes the number of permutations of n different things, taken r at a time

$$(n \in N\ r\ \in W, r \le n)$$

$${}^{n}P_r = n(n-1)(n-2) \ldots \ldots \ldots \ldots (n-r+1) = \frac{n!}{(n-r)!}$$

$${}^{n}P_n = n!,\ {}^{n}P_0 = 1\ {}^{n}P_1 = n$$

Number of arrangements of n distinct things taken all at a time = n!

${}^{n}P_r$ Is also denoted by A_r^n or P (n,r).

Combination:

A COMBINATION refers to each group or selection that can be formed by choosing some or all of the items without considering the order of the items within each group. Combinations typically involve problems related to selections, choices, the formation of distributed groups, committee formation, geometrical problems, and so on.

${}^{n}C_r$ Denotes the number of combinations of n different things taken r at a time

$$n \in N, r \in W, r \le n$$

$${}^{n}C_r = \frac{n!}{r!(n-r)!}$$

1. ${}^{n}C_r$ Is also denoted by or C (n, r). **2.** ${}^{n}P_r = {}^{n}C_r . r!$

Ex. How many three-digit numbers can be created using the digits 1, 2, 3, 4, 5, without repeating any digit? Additionally, how many of these three-digit numbers are even?

Sol. We need to fill three positions with five distinct objects.

Number of ways = ${}^{5}P_3 = 5 \times 4 \times 3 = 60$

For the 2nd part, unit digit can be filled in two ways & the remaining two digits can be filled in ${}^{4}P_2$ ways.

Number of even numbers = $2 \times {}^{4}P_2 = 24$.

Ex. Determine the exponent of 6 in the factorial of 50, denoted as 50!.

Sol.
$$E_2(50!) = \left[\frac{50}{2}\right] + \left[\frac{50}{4}\right] + \left[\frac{50}{8}\right] + \left[\frac{50}{16}\right] + \left[\frac{50}{32}\right] + \left[\frac{50}{64}\right]$$

(Where [ ] denotes integral part)

$$E_2(50!) = 25 + 12 + 6 + 3 + 1 + 0 = 47$$

$$E_3(50!) = \left[\frac{50}{3}\right] + \left[\frac{50}{9}\right] + \left[\frac{50}{27}\right] + \left[\frac{50}{81}\right]$$

$$E_3(50!) = 16 + 5 + 1 + 0 = 22$$

50! Can be written as $50! = 2^{47}.3^{22}\ldots\ldots\ldots$

Therefore exponent of 6 in 50! = 22

Ex. If (a) represents the number of permutations of (x + 2) items taken all at once, (b) denotes the number of permutations of (x) items taken 11 at a time, and (c) stands for the number of permutations of (x - 11) items taken all at once, such that (a = 182bc), then the value of (x) is.

Sol.

$$^{x+2}P_{x+2} = a$$
$$a = (x+2)!$$
$$^{x}P_{11} = b$$
$$b = \frac{x!}{(x-11)!}$$
$$^{x-11}P_{x-11} = c$$
$$c = (x-11)\,!$$
$$a = 182bc$$
$$(x+2)! = 182\frac{x!}{(x-11)!}(x-11)!$$
$$(x+2)(x+1) = 182 = 14 \times 13$$
$$x + 1 = 13$$
$$x = 12$$

Ex. The number of 4-letter words that can be created using the letters of the word 'ANSWER' and the count of words among them that start with a vowel are to be determined.

Sol. The count of ways to arrange 4 distinct letters chosen from a set of 6 distinct letters is.

$$^{6}C_{4}4! = \frac{6!}{2!} = 360$$

There are two vowels, A and E, in the word "ANSWER."

Total number of 4 letter words starting with A : A .... = $^{5}C_{3}3! = \frac{5!}{2!} = 60$

Total number of 4 letter words starting with E : E .... = $^{5}C_{3}3! = \frac{5!}{2!} = 60$

Total number of 4 letter words starting with a vowel= $60 + 60 = 120$.

Ex. $^{49}C_{3r-2} = {}^{49}C_{2r+1}$, find 'r'

Sol. $^{n}C_{r} = {}^{n}C_{s}$ if either r = s or r + s = n.

$$3r - 2 = 2r + 1$$
$$r = 3$$
$$3r - 2 + 2r + 1 = 49$$
$$5r - 1 = 49$$
$$r = 10$$
$$r = 3,10$$

Ex. Determine the position of the word 'QUEST' in dictionary order when all its letters are arranged in all possible ways.

Sol. Number of words beginning with E $= {}^{4}P_{4} = 24$

Number of words beginning with QE $= {}^{3}P_{3} = 6$

Number of words beginning with QS $= 6$

Number of words beginning with QT $= 6$.

Next word is 'QUEST'

Its rank is $24 + 6 + 6 + 6 + 1 = 43$.

Ex. Consider three coplanar parallel lines. Determine the maximum number of triangles that can be formed with vertices located at any set of \(p\) points on each of these lines.

Sol. The number of triangles with vertices on different lines = $^{P}C_{1} \times {}^{P}C_{1} \times {}^{P}C_{1} = P^{3}$

The number of triangles with two vertices on one line and the third vertex on any one of the other two lines

$$^{3}C_{1}\{\underbrace{^{P}C_{2}}_{\frac{p(p-1)}{2}} \times {}^{2P}C_{1}\} = 6p$$

So, the required number of triangles $= p^{3} + 3p^{2}(p-1) = p^{2}(4p-3)$

Ex. Selecting four points from a row of ten points in a way that none of them are consecutive can be done in how many different ways?

Sol. Total number of remaining non-selected points = 6

. . . . . .

Total number of gaps made by these 6 points = 6 + 1 = 7

Choosing 4 gaps from the 7 available gaps and placing points in those selected gaps will result in 4 points with the condition that none of them are consecutive.

x . . x . x . . x .

Total number of ways of selecting 4 gaps out of 7 gaps = $^7C_4$

Formation of groups

1. **(a)** The number of ways to divide (m + n) different things into two groups, where one group contains m things and the other has n things, is $\frac{(m+n)!}{m!n!}$ $(m \neq n)$.

(b) If m equals n, signifying equal groups, in this scenario, the number of divisions is $\frac{(2n)!}{n!n!2!}$. As it is possible to interchange the two groups in any one way without obtaining a new distribution.

(c) If 2n things are to be divided equally between two persons, then the number of ways: $\frac{(2n)!}{n!n!2!}$

2. **(a)** The number of ways in which (m + n + p) different things can be divided into three groups containing m, n, and p things, respectively, is: $\frac{(m+n+p)!}{m!n!p!}$ $m \neq n \neq p \ldots$.

(b) If m = n = p then the number of groups = $\frac{(3n)!}{n!n!n!3!}$

(c) If 3n items are to be distributed equally among three individuals, the number of ways in which it can be accomplished is: $\frac{(3n)!}{(n!)^3}$

3. Generally, the number of ways to distribute n distinct objects into l groups, with each group containing p objects, and m groups, with each containing q objects, is given by:

$$\frac{n!(\ell+m)!}{(p!)^{\ell}(q!)^{m}\ell!m!}$$

$$lp + mq = n$$

Ex. 12 different toys are to be distributed equally among three children. In how many ways can this be accomplished?

Sol. The task is to distribute 12 distinct items into three separate groups.

The "number of ways" = $\frac{12!}{4!4!4!} = 34650$

Ex. How many arrangements are there for dividing 15 students into 3 groups, each consisting of 5 students, while ensuring that two specific students are always together? Additionally, determine the number of arrangements if these groups are assigned to three different colleges.

Sol. Considering the two specific students as a single entity (as they are always together), the task is to form groups with 5 + 5 + 4 students from the total of 14 students.

Therefore total number of ways = $\frac{14!}{5!5!4!2!}$

Now, if these groups are intended to be sent to three different colleges, the total number of ways is

$$= \frac{14!}{5!5!4!2!} \times 3!$$

Ex. Determine the number of ways to distribute 52 cards among 4 players equally, ensuring that each player receives exactly one Ace.

Sol. Find the total number of ways to divide 48 cards (excluding the 4 Aces) into 4 groups.

$$\frac{48!}{(12!)^4 4!}$$

Now, distribute exactly one Ace to each group of 12 cards. Determine the total number of ways.

$$\frac{48!}{(12!)^4 4!} \times 4!$$

Now, distribute these groups of cards among four players.

$$\frac{48!}{(12!)^4 4!} \times 4!\, 4!$$

$$\frac{48!}{(12!)^4} \times 4!$$

Permutations under restrictions

(a) Number of permutations of n different objects taken r at a time, when a particular object is to be always included in each arrangement is r. n-1Pr-1 $^{n-1}P_{r-1}$

(b) Number of permutations of n different objects, taken r at a time, when a particular object is never taken in each arrangement is" $^{n-1}P_r$

(c) Number of permutations of n different objects taken all at a time, when m specified objects always occur (come) together is m! x (nm+ 1)!

(d) Number of permutations of n different objects, taken all at a time, when m specified objects never come together is n! m! x (nm+1)!