Divisions and distributions

DIVISORS

Let $N = p^0 . q^b . r^c$ .... where p, q, r........ are distinct primes & a, b, c....... are natural numbers then:

(a) The total number of divisors of (N), including 1 and (N), is

$$(a + 1)(b + 1)(c + 1)\ldots\ldots$$

(b) The sum of these divisors is

$$(p^0 + p^1 + p^2 + \ldots + p^a)(q^0 + q^1 + q^2 + \ldots + q^b)(r^0 + r^1 + r^2 + \ldots + r^c)$$

(c) The number of ways in which N can be resolved as a product of two factors is

$$\frac{1}{2}(a+1)(b+1)(c+1)\ \ldots\ldots \text{ if N is not a perfect square}$$

$$\frac{1}{2}[(a+1)(b+1)(c+1)\ \ldots\ldots + 1] \text{ if N is a perfect square}$$

(d) The number of ways a composite number N can be expressed as the product of two factors that are relatively prime (or coprime) is equal to 2^n–1, where n is the number of distinct prime factors in N.

1. Every natural number, except 1, has at least 2 divisors. If it has exactly two divisors, then it is termed as a prime number. The sequence of prime numbers begins with 2. All primes, except 2, are odd.
2. A number having more than 2 divisors is called a composite number. 2 is the only even number that is not composite.
3. Two natural numbers are considered relatively prime or coprime if their highest common factor (HCF) is one. For two natural numbers to be relatively prime, it is not necessary that one or both should be prime. They can both be composite but still coprime, for example, 4 and 25.
4. 1 is neither prime nor composite; however, it is coprime with every other natural number.
5. Two prime numbers are called twin prime numbers if their non-negative difference is 2. For example, 5 & 7, 19 & 17, etc.
6. All divisors, excluding 1 and the number itself, are referred to as proper divisors.

Ex. Determine the count of proper divisors for the number 38808 and calculate their sum.

Sol. 1. The number $38808 = 2^3 . 3^2 . 7^2 . 11$

Hence the total number of divisors (excluding 1 and itself i.e.38808)

$$= (3 + 1)(2 + 1)(2 + 1)(1 + 1) - 2 = 70$$

2. The sum of these divisors

$$(2^0 + 2^1 + 2^2 + 2^3)(3^0 + 3^1 + 3^2)(7^0 + 7^1 + 7^2)(11^0 + 11^1) - 1 - 38808$$

$$= (15)(13)(57)(12) - 1 - 38808$$

$$= 133380 - 1 - 38808$$

$$= 94571.$$

Ex.4 In how many ways can the number 18900 be expressed as a product of two factors that are relatively prime (coprime)?

Sol. Here, $N = 18900 = 2^2 . 3^3 . 5^2 . 7^1$

Number of different prime factors in 18900 = n = 4

Hence, number of ways in which 18900 can be resolved into two factors which are relative prime (or coprime) $= 2^{4-1} = 2^3 = 8$

TOTAL DISTRIBUTION

(a) Distribution of Distinct Objects:

The number of ways in which n distinct things can be distributed to p persons without any restriction on the number of things received by each person is denoted by (p^n).

(b) Distribution of Alike Objects:

The number of ways to distribute n identical things among p persons, allowing each person to receive none, one, or more things, is given by ${}^{n+p-1}C_{p-1}$

Ex. Determine the count of solutions for the equation $x + y + z = 6$, where x, y, z belong to the set of whole numbers.

Sol. Number of solutions = coefficient of x^6 in $(1 + x + x^2 + \ldots\ldots x^6)^3$

= coefficient of x^6 in $(1 - x^7)^3 (1 - x)^{-3}$

= coefficient of x^6 in $(1 - x)^{-3}$

${}^{3+6-1}C_6 = {}^8C_2 = 28.$

Ex. How many different ways can 5 distinct mangoes, 4 unique oranges, and 3 diverse apples be distributed among 3 children, ensuring that each child receives at least one mango?

Sol. The distribution of 5 distinct mangoes among 3 children, ensuring that each child receives at least 1, can occur in the following ways:

3 1 1

2 2 1

Total number of ways: $\left(\frac{5!}{3!1!1!2!} + \frac{5!}{2!2!2!}\right) \times 3!$

Now, the number of ways to distribute the remaining fruits (4 oranges + 3 apples) among 3 children is 3^7, as each fruit has 3 options.

Total number of ways $= \left(\frac{5!}{3!2!} + \frac{5!}{(2!)^3}\right) \times 3! \times 3^7$

Ex. Determine the count of non-negative integral solutions for the inequality $x + y + z \leq 20$.

Sol. Choose any number w, where $0 < w < 20$, and express the equation as:

$x + y + z + w = 20$ (here $x, y, z, w \geq 0$)

Total ways = ${}^{23}C_2$