APPLICATION OF MULTINOMIAL EXPANSION

Solution Of In Equation

1. The number of non - negative integral solution of $x_1 + x_2 + x_3 + \cdots \ldots \ldots \ldots + x_k = n$ is equal to the number of ways of distributing n identical objects among k groups which is equal ${}^{n+k-1}C_{k-1}$ or ${}^{n+k-1}C_n$ The same can also be obtained by equating the coefficient of x" in the expansion $(1 + x + x^2 + x^3 + \cdots \ldots..)^k$

$$= \text{coefficient of } x^n \text{ in } (1 - x)^{-k}$$

$$= \text{coefficient of } x^n \text{ in } (1 + {}^kC_1 x + {}^{k+1}C_2 x^2 + {}^{k+2}C_3 x^3 + \cdots \ldots.. + \cdots \ldots. + {}^{k+n-1}C_n x^n + \cdots \ldots \ldots..)$$

$$= {}^{n+k-1}C_n = {}^{n+k-1}C_{k-1}$$

Ex. Find the number of non-negative integral solution of the equation $x + y + z = 7$

Sol. The number of solutions is equal to the coefficient of x^7 in the expansion $(1 + x + x^2 + x^3 + x^4 \ldots)^3$

$$= \text{coefficient of } x^7 \text{ in } (\tfrac{1}{1-x})^3$$

$$= \text{coefficient of } x^7 \text{ in } (1 - x)^{-3}$$

$$= \text{coefficient of } x^7 \text{ in } (1 + {}^3C_1 x + {}^4C_2 x^2 + {}^5C_3 x^3 + {}^6C_4 x^4 + {}^7C_5 x^5 + {}^8C_6 x^6 + {}^9C_7 x^7 + \cdots \ldots \ldots)$$

$$= {}^9C_7 = \frac{9!}{2!7!}$$

$$= \frac{9\times 8}{2} = 36$$

2. The number of ways of choosing k objects out of p objects of 1st kind, q objects of 2nd kind, r objects of 3rd kind is equal to the coefficient of x in the expansion

$$(1 + x + x^2 + x^3 + \cdots \ldots \ldots + x^p)(1 + x + x^2 + x^3 + \cdots \ldots. x^q)(1 + x + x^2 + x^3 + \cdots \ldots. x^r)$$

If one object of each kind is to be included, then the number of ways of choosing k objects out of these objects is equal to the coefficient of x in the expansion of

$$(x + x^2 + x^3 + \cdots \ldots. x^p)(x + x^2 + x^3 + \cdots \ldots.. x^q)(x + x^2 + x^3 + \cdots \ldots.. x^r)$$

Ex. In how many ways can 10 identical toys be given to 3 children such that each receive at least one toy?

Sol. Let the three children A, B and C receive x, y, z toys

where $x \geq 1, y \geq 1, z \geq 1$ and $x + y + z = 10$

The required number of ways

$= \text{coefficient of } x^{10} \text{ in } (x + x^2 + x^3 + x^4 + \cdots \ldots \ldots.))^3$

$= \text{coefficient of } x^{10} \text{ in } x^3(1 - x)^{-3}$

$= \text{coefficient of } x^7 \text{ in } (1 + {}^3C_1 x + {}^4C_2 x^2 + {}^5C_3 x^3 + \cdots \ldots \ldots \ldots + {}^9C_7 x^7 + \cdots \ldots..)$

$= {}^9C_7 = 36$

3. The number of arrangements (permutations) of k objects out of p object of first kind, q objects of second kind and r objects of third kind is equal to the coefficient of x in the expansion of

$$k!\,(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \ldots \ldots \ldots + \frac{x^p}{p!})(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \ldots \ldots + \frac{x^q}{q!})(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \ldots + \frac{x^r}{r!})$$

Ex. Find the number of combinations and permutations of 4 letters taken from the word ARYABHATTA.

Sol. There are 10 letters A, A, A, A, TT, B, R, H, Y

The total number of combinations of 4 letters out of 4 A's, 2T and 4 different letters B, R, H, Y

$$= \text{coefficient of } x^4 \text{ in } (1 + x + x^2 + x^3 + x^4)(1 + x + x^2)(1 + x)^4$$

$$= \text{coefficient of } x^4 \text{ in } (1 + x + x^2 + x^3 + x^4)(1 + x + x^2)(1 + 4x + 6x^2 + 4x^3 + x^4)$$

$$= \text{coefficient of } x^4 \text{ in } (1 + 2x + 3x^2 + 3x^3 + 3x^4 + 2x^5 + x^6)(1 + 4x + 6x^2 + 4x^3 + x^4)$$

$$= 1 + 8 + 18 + 12 + 3 = 42$$

And the number of permutations

$$= \text{ coefficient of } x^4 \text{ in } 4!\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}\right)\left(1+\frac{x}{1!}\right)^4$$

$= \text{ coefficient of } x^4$ in

$$4!\left(1+x^2+\frac{x^4}{4}+2x+x^2+x^3+\frac{x^3}{3!}+\frac{x^4}{3!}+\frac{x^5}{3!2!}+\frac{x^4}{4!}+\frac{x^5}{4!}+\frac{x^6}{2!4!}\right)(1+4x+6x^2+4x^3+x^4)$$

$$= \text{ coefficient of } x^4 \text{ in } 4!\left(1+2x+2x^2+\frac{7x^3}{6}+\frac{11}{24}x^4+\cdots\ldots\ldots\right)(1+4x+6x^2+4x^3+x^4)$$

$$= 4!\left(1+8+12+\frac{14}{3}+\frac{11}{24}\right) = 504+112+11 = 627$$

OR

There are 10 letters A, A, A, A, T, T, B, R, H, Y

We have following cases

Case 1. When all are different i.e. # Δ □ O. 4 letters out of 6 letters can be selected in ${}^6C_4$ ways and then 4 letters can be arranged in 4 ! ways.

$${}^6C_4 \times 4\,!$$

Case 2. When two identical and two different ## ΔO

Which can be performed in ${}^2C_1 \cdot {}^5C_2 \cdot \frac{4!}{2!}$ ways

Case 3. When three identical # # # Δ

$${}^1C_1 \times {}^5C_1 \times \frac{4!}{3!}$$

Case 4. When two are alike and two are also alike i.e. # #OO

$${}^2C_2 \times \frac{4!}{2!2!}$$

Case 5. When all are identical

$${}^1C_1 \cdot \frac{4!}{4!}$$

For combination of 4 letters ${}^6C_4 + {}^2C_1 \cdot {}^5C_2 + {}^1C_1 \cdot {}^5C_1 + {}^2C_2 + {}^1C_1 = 42$

For permutation ${}^6C_4 \cdot 4! + {}^2C_1 \cdot {}^5C_2 \cdot \frac{4!}{2!} + {}^1C_1 \cdot {}^5C_1 \cdot \frac{4!}{3!} + {}^2C_2 \cdot \frac{4!}{2!2!} + {}^1C_1 \cdot \frac{4!}{4!} = 627$