PROPERTIES OF BINOMIAL COEFFICIENTS

$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \cdots \ldots \ldots + C_nx^n = \sum_{r=0}^{n} {}^nC_r r^r; n \in N$ .... (i)

Where $C_0, C_1, C_2, \ldots \ldots \ldots \ldots C_r$ are called **combinatorial (Binomial) coefficients**.

(a) The sum of all the Binomial coefficients is 2^n.

Put $x = 1$, in (i)

We get $C_0 + C_1 + C_2 + \cdots \ldots \ldots \ldots + C_n = 2^n$

$\Rightarrow \sum_{r=0}^{n} {}^nC_r = 0$ .... (ii)

(b) Put x= –1 in (i)

We get $C_0 - C_1 + C_2 - C_3 \ldots \ldots \ldots \ldots + C_n = 0$

$\Rightarrow \sum_{r=0}^{n}(-1)^{rn}C_r = 0$ ... (iii)

(c) The sum of the Binomial coefficients at odd position is equal to the sum of the Binomial coefficients at even position and each is equal to 2^{n-1}.

From (ii) & (iii),

$$C_0 + C_2 + C_4 + \cdots \ldots \ldots . . = C_1 + C_3 + C_5 + \cdots \ldots = 2^{n-1}$$

(d) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(e) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

(f) ${}^nC_r = \frac{n}{r}\ {}^{n-1}C_{r-1} = \frac{n}{r}\cdot\frac{n-1}{r-1}\ {}^{n-2}C_{r-2} = \cdots \ldots . . = \frac{n(n-1)(n-2)\ldots.(n-r+1)}{r(r-1)(r-2)\ldots..1}$

(g) ${}^nC_r = \frac{r+1}{n+1}\cdot\ {}^{n+1}C_{r+1}$

Ex. Prove that: ${}^{25}C_{10} + {}^{24}C_{10} + \cdots \ldots \ldots + {}^{10}C_{10} = {}^{26}C_{11}$

Sol. LHS ${}^{10}C_{10} + {}^{11}C_{10} + {}^{12}C_{10} + \cdots \ldots \ldots \ldots . + {}^{25}C_{10}$

$${}^{11}C_{11} + {}^{11}C_{10} + {}^{12}C_{10} + \cdots \ldots . + {}^{25}C_{10}$$

$${}^{12}C_{11} + {}^{12}C_{10} + \cdots \ldots . . + {}^{25}C_{10}$$

$${}^{13}C_{11} + {}^{13}C_{10} + \cdots \ldots . . .\ {}^{25}C_{10}$$

And so on. RHS $= {}^{26}C_{11}$

Alternative

LHS = coefficient of x^{10} in $\{(1+x)^{10}_m + (1+x)^{11} + \cdots \ldots \ldots \ldots \ldots . . . (1+x)^{25}\}$

$$\text{Coefficient of } x^{10} \text{ in } [(1+x)^{10}\frac{\{1+x\}^{16}-1}{1+x-1}]$$

$$\text{Coefficient of } x^{10} \text{ in } \frac{[(1+x)^{26}-(1+x)^{10}]}{x}$$

$$\text{Coefficient of } x^{10} \text{ in } [(1+x)^{26} - (1+x)^{10}]$$

$$= {}^{26}C_{11} - 0 = {}^{26}C_{11}$$

Ex. Prove that:

(a) $C_1 + 2C_2 + 3C_3 + \cdots \ldots . + {}^nC_n = n \cdot 2^{n-1}$

(b) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots \ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

Sol. (a) L.H.S. $= \sum_{r=1}^{n} r \cdot {}^nC_r = \sum_{r=1}^{n} r \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1}$

$$\sum_{r=1}^{n} r \cdot {}^{n-1}C_{r-1} = n \cdot [{}^{n-1}C_0 + {}^{n-1}C_0 + \cdots . . + {}^{n-1}C_{n-1}]$$

$$= n \cdot 2^{n-1}$$

Aliter: (Using method of differentiation)

$$(1+x)^n_m = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \cdots \ldots . + {}^nC_nx^n \quad \text{.......... (A)}$$

Differentiating (A),

We get $n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \cdots \ldots . + n \cdot C_nx^{n-1}$.

Put $x = 1$,

$$C_1 + 2C_2 + 3C_3 + \cdots \ldots \ldots + n \cdot C_n = n \cdot 2^{n-1}$$

(b) L.H.S. $= \sum_{r=0}^{n} \frac{C_r}{r+1} = \frac{1}{n+1}\sum_{r=0}^{n} \frac{n+1}{r+1}\ {}^{n}C_r$

$= \frac{1}{n+1}\sum_{r=0}^{n}\ {}^{n+1}C_{r+1}$

$= \frac{1}{n+1}[\ {}^{n+1}C_1 + \ {}^{n+1}C_2 + \cdots . + {}^{n+1}C_{n+1}]$

$= \frac{1}{n+1}[2^{n+1} - 1]$

Aliter: (Using method of integration)

Integrating (A),

We get $\frac{(1+x)^{n+1}}{n+1} + C = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \cdots \ldots + \frac{C_nx^{n+1}}{n+1}$

(Where C is a constant)

Put $x = 0$

We get, $C = -\frac{1}{-n+1}$

$$\frac{(1+x)^{n+1}-1}{n+1} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \cdots \ldots + \frac{C_nx^{n+1}}{n+1}$$

Put $x = 1,$

We get $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots \ldots \cdot \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

Put $x = -1,$

We get $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \cdots \ldots = \frac{1}{n+1}$

Ex. Prove that $C_1 - C_3 + C_5 - \cdots \ldots . . = 2^{\frac{n}{2}}\sin \cdot \frac{n\pi}{4}$

Sol. Consider the expansion

$$(1+x)^n_m = C_0 + C_1x + C_2x^2 + \cdots \ldots + C_nx^n \ldots . \text{(i)}$$

Putting x = – I, in (i)

We get $(1-i)^n = C_0 - C_1i - C_2 + C_3i + C_4 + \cdots \ldots . (-1)^nC_ni^n$

$$2^{\frac{n}{2}}\left[\cos\left(-\frac{n\pi}{4}\right) + i\sin\left(-\frac{n\pi}{4}\right)\right]$$

$$= (C_0 - C_2 + C_4 - \cdots \ldots .) - i(C_1 - C_3 + C_5 - \cdots \ldots) \ldots . . \text{(ii)}$$

Equating the imaginary part in (ii)

We get $C_1 - C_3 + C_5 - \cdots \ldots \ldots = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}$

Ex. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots . . + C_nx^n$ then prove that

$$\sum_{0\le i<j\le n}(C_i + C_j)^2 = (n-1)^{2n}C_n + 2^{2n}$$

Sol. L.H.S $\sum_{0\le i<j\le n}(C_i + C_j)^2$

$$= (C_0 + C_1)^2 + (C_0 + C_2)^2 + \cdots + (C_0 + C_n)^2 + (C_1 + C_2)^2 + (C_1 + C_3)^2 + \cdots$$
$$+ (C_1 + C_n)^2 + (C_2 + C_3)^2 + (C_2 + C_4)^2 + \cdots + (C_2 + C_n)^2 + \cdots . + (C_{n-1} + C_n)^2$$

$$= n(C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2) + 2\sum_{0<i<i<n} C_iC_j$$

$$= n^{2n}C_n + 2\cdot\left\{2^{2n-1} - \frac{2n!}{2\cdot n!\,n!}\right\}$$

$$n.\ {}^{2n}C_n + 2^{2n} - {}^{2n}C_n$$

$$= (n-1)\cdot\ {}^{2n}C_n + 2^{2n} = \text{R.H.S}$$