IMPORTANT TERMS IN THE BINOMIAL EXPANSION

1. General Term:

The general term or the $(r+1)^{th}$term in the expansion of $(x + y)^n$ is given by

$$Tr+1 = {}^nC_n x^{n-r} y^r$$

Ex. Find

(a) 28^{th} term of $(5x+8y)^{30}$ (b) 7^{th} term of $(\frac{4x}{5}-\frac{5}{2x})^9$

Sol. (a) $T_{27+1} = {}^{30}C_{27}(5x)^{30-27}(8y)^{27}$

$$= \frac{30!}{3!27!}(5x)^3 \cdot (8y)^{27}$$

(b) 7^{th} term of $(\frac{4x}{5}-\frac{5}{2x})^9$

$$T_6 + 1 = {}^9C_6(\frac{4x}{5})^{9-6}(-\frac{5}{2x})^6$$

$$= \frac{9!}{3!6!}\left(\frac{4x}{5}\right)^3\left(\frac{5}{2x}\right)^6$$

$$= \frac{10500}{x^3}$$

2. Middle Term:

The term or terms at the center of the expansion of $(x+y)^n$ is (are):

(a) If n is even, there exists only one central term, and it is determined by:

$$T_{\frac{(n+2)}{2}} = C^n_{\frac{n}{2}} \cdot x^{\frac{n}{2}} \cdot y^{\frac{n}{2}}$$

(b) If n is odd, there are two central terms, which are:

$$T_{\frac{(n+1)}{2}} \& T_{\frac{(n+1)}{2}+1}$$

The central term possesses the highest binomial coefficient, and in the case of two central terms, their coefficients will be identical. ${}^nC_r$ will be maximum

When $r = \frac{n}{2}$ if n is even

When $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$ if n is odd

In the expansion of $(1+x)^n$ the term with the highest binomial coefficient will be the central term.

Ex. Determine the central term or terms in the expansion of

(a) $\left(1-\frac{x^2}{2}\right)^{14}$ (b) $\left(3a-\frac{a^3}{6}\right)^9$

Sol. (a) $\left(1-\frac{x^2}{2}\right)^{14}$ (b) $\left(1-\frac{x^2}{2}\right)^{14}$

Here, n is even, therefore middle term is $\left(\frac{14+2}{2}\right)^{th}$ term.

It means T_8 is middle term

$$T_s = C^{14}_7\left(-\frac{x^2}{2}\right)^7 = -\frac{429}{16}x^{14}$$

(b) $\left(3a - \frac{a^3}{6}\right)^9$

Here, n is odd therefore, middle terms are $(\frac{9+1}{2})^{th}$ &$(\frac{9+1}{2}+1)^{th}$

It means T_5 & T_6 is middle terms

$$T_5 = C_4^9(3a)^{9-4}\left(-\frac{a^3}{6}\right)^4 = \frac{189}{8}a^{17}$$

$$T_6 = C_5^9(3a)^{9-5}(-\frac{a^3}{6})^5 = -\frac{21}{16}a^{19}$$

3. Term Independent of x :

A term that is independent of x does not contain x. Therefore, determine the value of r for which the exponent of x is zero.

Ex. Find the term independent of x in $\left[\sqrt{\frac{x}{3}} + \sqrt{\left(\frac{3}{2x^2}\right)}\right]^{10}$.

Sol. General term in the expansion is

$$C_r^{10}\ \left(\frac{x}{3}\right)^{\frac{r}{2}}\left(\frac{3}{2x^2}\right)^{\frac{10-r}{2}}$$

$$C_r^{10}\ x^{\frac{3r}{2}-10}\cdot\frac{3^{5-r}}{2^{\frac{10-r}{2}}}$$

For constant term,

$$r = \frac{10}{3}$$

Which is not an integer. Therefore, there will be no constant term.

(d) Numerically Greatest Term:

The binomial expansion of $(a + b)^n$ is expressed as follows: –

$$(a+b)^n = {}^nC_0a^nb^0 + {}^nC_1a^{n-1}b_1 + {}^nC_2a^{n-2}b_2 + \cdots + {}^nC_ra^{n-r}b^r + \cdots \ldots + {}^nC_na^0b_n$$

$$(a+b)_{n^n} = {}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^2 + {}^nC_2a^{n-2}b^2 + \cdots + {}^nC_ra^{n-r}b^r + \cdots \ldots + {}^nC_na^0b^n$$

By substituting specific values for a, and b on the right-hand side (RHS), each term in the binomial expansion will assume a particular value. The term with the numerically highest value is referred to as the numerically greatest term.

T_r and T_{r+1} be the r^{th} and $(r+1)^{th}$ Terms respectively

$$T_r = {}^nC_{r-1}a^{n-(r-1)}b^{r-1}$$

$$T_{r+41} = n_{C_r}a^{n-r}b^r$$

$$\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^nC_r}{{}^nC_{r-1}}\frac{a^{n-r}b^r}{a^{n-r+1}b^{r-1}}\right| = \frac{n-r+1}{r}\cdot\left|\frac{b}{a}\right|$$

$$\left|\frac{T_{r+1}}{T_r}\right| \geq 1$$

$$\left(\frac{n-r+1}{r}\right)\left|\frac{b}{a}\right| \geq 1$$

$$\frac{n+l}{r} - 1 \geq \left|\frac{a}{b}\right|$$

$$r \leq \frac{n+1}{1+\left|\frac{a}{b}\right|}$$

Case I: When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer (say m),

(a) $T_{r+1} > T_{mr}$

When, $r < m(r = 1,2,3 \ldots., m-1)$

$$T_2 > T_1, T_3 > T_2 \ldots\ldots. T_m > T_{m-1}$$

(b) $T_{r+1} = T_r$

When, $r = m$

$$T_{m+1} = T_m$$

(c) $T_{r+1} < T_r$

$$r > m(r = m+1, m+2, \ldots \ldots \ldots..n)$$

$$T_{m+2} < T_{m+\text{ qund}}\ T_{m+3} < T_{m+2}, \ldots \ldots \ldots..T_{n+1} < T_n$$

Conclusion

When $\frac{n+1}{1+|\frac{a}{b}|}$ is an integer, say m, then T_m and T_{m+1} will be numerically greatest terms (both terms are equal in magnitude)

Case - II When $\frac{n+1}{1+|\frac{a}{b}|}$ is not an integer (Let its integral part be m),

(a)

$$T_{r+1} > T_r$$

$$r < \frac{n+1}{1+|\frac{a}{b}|} (r = 1,2,3, \ldots \ldots \ldots, m-1, m)$$

$$T_2 > T_{mme}T_3 > T_2, \ldots \ldots \ldots \ldots \ldots T_{m+1} > T_m$$

(b)

$$T_{r+1} < T_r$$

$$r > \frac{n+1}{1+|\frac{a}{b}|} (r = m+1, m+2, \ldots \ldots \ldots n)$$

$$T_{m+2} < T_{m+1}\; T_{m+3} < T_{m+2}, \ldots \ldots \ldots \ldots .., T_{n+1} < T_n$$

Conclusion

When $\frac{n+1}{1+|\frac{a}{b}|}$ not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

1. In any binomial expansion, the central term or terms possess the highest binomial coefficient.
   In the expansion of $(a + b)^n$
2. In order to obtain the term having numerically greatest coefficient, put $a = b = 1$, and proceed as discussed above.

n	No. of Greatest Binomial Coefficient	Greatest Binomial Coefficient
Even	1	${}^nC_{\frac{n}{2}}$
Odd	2	${}^nC_{\frac{(n-1)}{2}}$ and ${}^nC_{\frac{(n+1)}{2}}$ (Values of both these coefficients are equal)

Ex. Determine the numerically greatest term in the expansion of $(3 - 5x)^{11}$ when x is equal to?

Sol. Using

$$\frac{n+1}{1+|\frac{a}{b}|} - 1 \le r \le \frac{n+1}{1+|\frac{a}{b}|}$$

$$\frac{11+1}{1+|\frac{3}{-5x}|} - 1 \le r \le \frac{11+1}{1+|\frac{3}{-5x}|}$$

Solving we get $2 < r < 3$

$r = 2, 3$

So, the greatest terms are T_{2+1} and T_{3+1}.

Greatest term (when r = 2)

$$T_3 = {}^{11}C_2 \cdot 3^9(-5x)^2 = 55.3^9 = T_4$$

From above we say that the value of both greatest terms are equal.

Ex. For a positive integer n, demonstrate that the integral part of $(7 + 4\sqrt{3})^n$ is an odd number.

Sol. Let $(7 + 4\sqrt{3})^n = I + f$ ... (i)

Where I & f are its integral and fractional parts respectively.

It means $0 < f < 1$

Now $0 < 7 - 4\sqrt{3} < 1$

$$0 < (7 - 4\sqrt{3})^n < 1$$

Let $(7 - 4\sqrt{3})^n = f'$ ... (ii)

$$0 < f' < 1$$

Adding (i) and (ii)

$$I + f + f' = (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n$$

$$2[\,^nC_0 7^n + \,^nC_2 7^{n-2}(4\sqrt{3})^2 + \cdots ..]$$

I+ f+ f' = even integer

(f +f' must be an integer)

$$0 < f + f' < 2$$

$$f + f' = 1$$

I+1=even integer therefore I is an odd integer.

Ex. What is the remainder when dividing 5^{99} by 13?

Sol. $5^{99} = 5.5^{98} = 5 \cdot (25)^{49} = 5(26-1)^{49}$

$$5[\,^{49}C_0(26)^{49} - \,^{49}C_1(26)^{48} + \cdots \ldots \ldots . + \,^{49}C_{48}(26)^1 - \,^{49}C_{49}(26)^0]$$

$$5[\,^{49}C_0(26)^{49} - \,^{49}C_1(26)^{48} + \cdots \ldots \ldots .. + \,^{49}C_{48}(26)^1 - 1]$$

$$5[\,^{49}C_0(26))^{49} - \,^{49}C_1(26))^{48} + \cdots \ldots \ldots . + \,^{49}C_{48}(26)^1 - 13] + 60$$

$$13(k) + 52 + 8 \text{ (where k is a positive integer)}$$

$$13(k+4) + 8$$

Hence, remainder is 8.

Some Standard Expansions

1. Consider the expansion

$$(x + y)^n = \sum_{r=0}^{n} \,^nC_r x^{n-r} y^r$$

$$= \,^nC_0 x^n y^0 + \,^nC_1 x^{n-1} y^1 + \cdots \ldots \ldots .. + \,^nC_r x^{n-r} y^r + \cdots \ldots \ldots . + \,^nC_n x^0 y^n \ldots .. \text{(i)}$$

2. Now replace y – y we get

$$(x - y)^n = \sum_{r=0}^{n} \,^nC_r (-1)^r x^{n-r} y^r$$

$$= \,^nC_0 x^n y^0 - \,^nC_1 x^{n-1} y^1 + \cdots + \,^nC_r (-1)^r x^{n-r} y^r + \cdots + \,^nC_n (-1)^n x^0 y^n \text{........ (ii)}$$

3. Adding 1. &2, we get

$$(x + y)^n + (x - y)^n = 2[\,^nC_0 x^n y^0 + \,^nC_2 x^{n-2} y^2 + \cdots \ldots \ldots ..]$$

4. Subtracting (ii) from (i), we get

$$(x + y)^n - (x - y)^n = 2[\,^nC_1 x^{n-1} y^1 + \,^nC_3 x^{n-3} y^3 + \cdots \ldots ...]$$