Chapter 7

Binomial Theorem

INTRODUCTION

The binomial theorem is the method of expanding an expression that has been raised to any finite power. A binomial theorem is a powerful tool of expansion which has applications in Algebra, probability, etc.

BINOMIAL EXPRESSION

Any algebraic expression which contains two dissimilar terms is called **Binomial expression.**

For example: $x - y, xy + \frac{1}{x}, \frac{1}{z} - 1, \frac{1}{(x-y)^{\frac{1}{3}}} + 3$ etc.

Terminology Used in Binomial Theorem

Factorial notation: $\underline{|n}$ or n! is pronounced as factorial n and is defined as

$$n! = \begin{cases} n(n-1)(n-2)\ldots\ldots 3.2.1 & \text{if } n \in N \\ 1 & \text{if } n = 0 \end{cases}$$

$$vn! = n \cdot (n-1)!; n \in N$$

BINOMIAL THEOREM

The formula by which any positive integral power of a Binomial expression can be expanded in the form of a series is known as **BINOMIAL THEOREM.**

If x, y R and nN, then:

$$(x + y)^n = {}^nC_0x^n + n_{c_1}x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots + {}^nC_1x^{n-r}y^r + nC_ny_n^n$$

This theorem can be proved by induction.

(a)	The number of terms in the expansion is (n+1) i.e. one more than the index.
(b)	The sum of the indices of x & y in each term is n.
(c)	The Binomial coefficients of the terms (${}^nC_0$, ${}^nC_1$) equidistant from the beginning and the end are equal. I.e. ${}^nC_P = {}^nC_{n-r}$
(d)	Symbol ${}^nC_r$ can also be denoted by. $\binom{n}{r}$, C(n, r)
	The coefficient of x^r in $(1 + x)^n = n_{C_r}$ & that in $(1 - x)^n = (-1)^r \cdot {}^nr$

Some Important Expansions:

1. $(1 + x)_m^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots {}^nC_nx^n$

2. $(1 - x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 + \cdots \ldots\ldots + (-1)^n . {}^nC_nx^n$

Ex. Expand the following Binomials:

(a) $(x - 3)^5$ (b) $(1 - \frac{3x^2}{2})^4$

Sol. (a) $(x - 3)^5$

$${}^5C_0x^5 + {}^5C_1x^4(-3)^1 + {}^5C_2x^3(-3)^2 + {}^5C_3x^2(-3)^3 + {}^5C_4x(-3)^4 + {}^5C_5(-3)^5$$

$$x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$$

(b) $(1-\frac{3x^2}{2})^4$

$$C_0^4 + C_1^4(\frac{-3x^2}{2}) + C_2^4(\frac{-3x^2}{2})^2 + C_3^4(\frac{-3x^2}{2})^3 + C_4^4(\frac{-3x^2}{2})^4$$
$$= 1 - 6x^2 + \frac{27}{2}x^4 - \frac{27}{2}x^6 + \frac{81}{16}x^8$$

Ex. Find the value of $\frac{(18^3+7^3+3\times18\times7\times25)}{3^6+6\times243\times2+15\times81\times4+20\times27\times8+15\times9\times16+6\times3\times32+64}$

Sol. The numerator is of the form

$$a^3 + b^3 + 3ab(a+b) = (a+b)^3$$

Where, $a = 18$ and $b = 7$

$$N^r = (18+7)^3 = (25)^3$$

Denominator can be written as

$$\frac{Nr}{Dr} = \frac{(25)^3}{(25)^3} = 1$$

BINOMIAL THEOREM FOR POSITIVE INDEX

The Binomial Theorem is a mathematical formula enabling the expansion of any power of a binomial expression into a series. When dealing with a positive integer (n), the expansion is expressed as follows:

$$(a+x)^n$$
$${}^nC_0a^n + {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 + \cdots + {}^nC_ra^{n-r}x^r + \cdots + {}^nC_nx^n$$
$$\sum_{r=0}^{n} {}^nC_ra^{n-r}x^r$$

Where ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are called **Binomial co-efficient.**

Similarly

$$(a-x)^n = {}^nC_0a^n - {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 - \cdots + (-1)^mC_ra^{n-r}x^r + \cdots + (-1)^{nn}C_nX^n$$
$$(a-x)^n_{\sim\sim} = \sum_{r=0}^{n}(-1)^{rn}C_ra^{n-r}x^r$$

Replacing $a = 1$

We get $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_rX^r + \cdots + {}^nC_nX^n$

And $(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2X^2 - \cdots + (-1)^{rn}C_rx^r + \cdots + (-1)^{nn}C_nx^n$

Observations:

- There exist (n+1) terms in the expansion of$(a+x)^n$..
- In the expansion of $(a+x)^n$ the sum of the powers of (x) and (a) in each term remains constant and is equal to (n).
- The general term in the expansion of $(a+x)^n$ is denoted by the (r+1)th term, and it is represented as:

$$T_{r+1} = {}^nC_ra^{n-r}X^r$$

- The term located at position p from the end is equivalent to the term at position (n−p+2) from the beginning in the expansion of$(a+x)^n$.
- Coefficient of x^r in expansion of $(a+x)^n$ is ${}^nC_ra^{n-r}x^r$.

$${}^nC_x = {}^nC_y$$
$$x = y \text{ or } x + y = n.$$

In the expansion of $(a+x)^n$ and $(a-x)^n, x^r$ occurs in $(r+1)^{th}$ term.

Ex. If the coefficients corresponding to the second, third, and fourth terms in the expansion of the binomial expression are...

$(1+x)^n$ Are in A.P., show that $n = 7$.

Sol. According to the question ${}^nC_1 \cdot {}^nC_2 \cdot {}^nC_3$are in A.P.

$$\frac{2n(n-1)}{2} = n + \frac{n(n-1)(n-2)}{6}$$
$$n^2 - 9n + 14 = 0$$
$$n-2)(n-7) = 0$$
$$n = 2 \text{ or } 7$$

Since, the symbol ${}^nC_3$ demands that n should be ≥ 3, n cannot be 2,

$n = 7$ only..

Ex. Find the

(a) Last digit (b) Last two digit (c) Last three digit of 17^{256}.

Sol. $(17)^{256} = (289)^{128} = (290-1)^{128}$

$$[\,{}^{128}C_0(290)^{128} - {}^{128}C_1(290)^{127} + \cdots \ldots\ldots\ldots] + {}^{128}C_{126}(290)^2 - {}^{128}C_{127}(290) + 1$$

$$= 1000m + {}^{128}C_{126}(290)^2 - {}^{128}C_{127}(290) + 1$$

$$= 1000\text{ m} + \frac{128\times127}{2}\times(290)^2 - \frac{128\times290}{1} + 1$$

$$1000m + 683527680 + 1$$

Hence, the last digit is 1. Last two digits is 81. Last three digit is 681.

Ex. If the binomial coefficients for the, $(2r+4)^{th}$ and $(r-2)^{th}$ terms in the expansion of $(a+bx)^{18}$ are equal find r.

Sol. This is possible only when

Either $2r + 3 = r - 3$ ... (1)

Or $2r + 3 + r - 3 = 18$ ... (2)

From (1) $r = -6$ not possible but from (2) $r = 6$

Hence $r = 6$ is the only solution.

Ex. Determine the coefficient of

(a) x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$, (b) x^{-7} in $\left(ax - \frac{1}{b^2}\right)^{11}$.

Determine the relationship between a, and b when these coefficients are equal.

Sol. The general term in $(ax^2 + \frac{1}{bx})^{11} = {}^{11}C_r(ax^2)^{11-r}(\frac{1}{bx})^r$

$$= {}^{11}C_r\frac{a^{11-r}}{b^r}x^{22-3r}$$

If in this term power of x is 7, then $22 - 3r = 7$

$$r = 5$$

Coefficient of $x^7 = {}^{11}C_5\frac{a^6}{b^5}$ ... (1)

The general term in $(ax - \frac{1}{bx^2})^{11} = (-1)^r\,{}^{11}C_r(ax)^{11-r}(\frac{1}{bx^2})^r$

$$(-1)^r\,{}^{11}C_r\frac{a^{11-r}}{b^r}x^{11-3r}$$

If in this term power of x is –7, then $11 - 3r = -7 \Rightarrow r = 6$

Coefficient of $x^{-7} = (-1)^6\,{}^{11}C_6\frac{a^{11-6}}{b^6} = {}^{11}C_5\frac{a^5}{b^6}$

If these two coefficient are equal,

$$ {}^{11}C_5\frac{a^6}{b^5} = {}^{11}C_5\frac{a^5}{b^6}$$

$$a^6b^6 = a^5b^5$$

$$a^5b^5(ab-1) = 0$$

$$ab = 1(a \neq 0, b \neq 0)$$