PROPERTIES OF MODULUS OF A COMPLEX NUMBER

The following properties regarding the modulus of complex numbers are valid:

1. $|z| = 0$ iff $z = 0$ and $|z| > 0$ iff $z \neq 0$.

2. $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq \text{lm}(z) \leq |z|$

3. $|z_1 z_2| = |z_1||z_2|$

In general $|z_1 z_2 z_3 \dots . z_n| = |z_1||z_2||z_3| \dots |z_n|$

4. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, where $z_2 \neq 0$

5. $|z_1 \pm z_2|^2 = |z_2|^2 + |z_1|^2 \pm 2\,\text{Re}(z_1\bar{z}_2) = |z_1|^2 + |z_2|^2 \pm 2\,\text{Re}(\bar{z}_1 \cdot z_2)$

We have,

$$|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\overline{z_1 \pm z_2})$$
$$= (z_1 \pm z_2)(\bar{z}_1 + \bar{z}_2)$$
$$= z_1\bar{z}_1 \pm z_1\bar{z}_2 \pm \bar{z}_1 z_2 + z_2\bar{z}_2$$
$$= |z_1|^2 + |z_2|^2 \pm \left(z_1\bar{z}_2 + \overline{z_1\bar{z}_2}\right)$$
$$= |z_1|^2 + |z_2|^2 \pm 2\text{Re}(z_1 \cdot \bar{z}_2)$$

6. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

7. $|z_1 + z_2| \leq |z_1| + |z_2|$

8. $|z_1 - z_2| \geq ||z_1| - |z_2||$

9. $||z_1| - |z_2|| \leq |z_1| + |z_2|$

10. $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \frac{\pi}{2}$

11. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary number.

12. If $|z_1| \leq 1$ and $|z_2| \leq 1$, then

$$|z_1 - z_2|^2 \leq (|z_1| - |z_2|)^2 + (\arg z_1 - \arg z_2)^2 \text{ and}$$
$$|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2 - (\arg z_1 - \arg z_2)^2$$
$$|z_1 + z_2 + \cdots . + z_n| = |\bar{z}_1 + \bar{z}_2 + \cdots .. + \bar{z}_n|$$

The minimum value of $|z - z_1| + |z - z_2| = |z_1 - z_2|$

$$|z|^2 = z\bar{z}$$