Definition of a Complex Number

Real and Imaginary Part

The square root of a negative number is referred to as an imaginary number. When solving equations like $x^2 + 1 = 0$, a quantity $\sqrt{-1}$ is obtaind and denoted as i (iota), representing an imaginary unit. Additionally, $\sqrt{-2}$ is an imaginary number and can be expressed as follows:

$$\sqrt{-2} = \sqrt{2} \times \sqrt{-1} = \sqrt{2}i$$

$$a < 0, \text{ then } \sqrt{a} = \sqrt{|a|}j$$

Integral Powers of i

We have $i = \sqrt{-1}$ so $i^2 = -1, i^3 = -I, i^4 = 1$

For any $n \in N$,

We have $i^{4n+1} = i, \quad i^{4n+2} = -1$

$i^{4n+3} = -i, \quad i^{4n} = 1$

Thus any integral power of i can be expressed as ± 1 or $\pm i$.

In other words. $i^n \begin{cases} (-1)^{\frac{n}{2}} \text{ if n is even integer} \\ (-i)^{\frac{n-1}{2}}; \text{ if nis odd integer} \end{cases}$

Also $i^{-n} = \frac{1}{i^n}$

Ex. Evaluate:

1. j^{786} 2. $(-\sqrt{-1})^{23}$ 3. $\frac{i^2+j^3+j^4+j^5}{i+j^2+i^3}$

Sol.
1. $\mathbf{i^{786} = i^{4\times196+2} = i^2 = -1}$
2. $\mathbf{(-\sqrt{-1})^{23} = (-1 \times i)^{23} = (-i)^{23} = -(i)^{23} = -(i)^{4\times5+3} = -i^3 = -(-i) = i}$
3. $\mathbf{\frac{i^2+j^3+j^4+j^5}{i+j^2+i^3} = \frac{i^2(1+i+j^2+j^3)}{i+j^2+j^3} = \frac{(-1)(1+i-1-i)}{i-1-i} = \frac{0}{-1} = 0}$

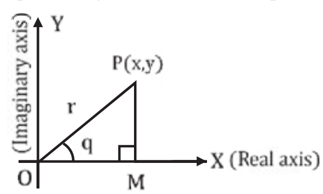
$\sqrt{\mathbf{a}} \times \sqrt{\mathbf{b}} = \sqrt{\mathbf{ab}}$ Is true iff at least one of a, and b are non-negative. If $a < 0$ and $b < 0$, then

$$\sqrt{a} \times \sqrt{b} = \sqrt{-|a|} \times \sqrt{-|b|}$$

$$i\sqrt{|a|} \times i\sqrt{|b|} = -\sqrt{ab}$$

Representation of a Complex Number

The complex number $z = x + iy$ can be correlated with the ordered pair $P(x, y)$. In this context, we envision two perpendicular lines, OX and OY, akin to the Cartesian system, denoted as the real axis and imaginary axis, respectively. Here, O represents the origin of reference. The resulting plane is termed **the Argand plane, Gaussian plane, or complex plane.** The complex number z is depicted by the point P corresponding to the ordered pair P is referred to as the affix of z.



Modulus of a Complex Number

The modulus of a complex number $z = x + iy$ is defined as

$$|z| = \sqrt{x^2 + y^2} = \sqrt{\{Re(z)\}^2 + \{lm(z)\}^2}.$$

The distance of z from the origin when represented on the Argand plane. It is denoted by mod (z) or $|z|$, or r.

Here $OP = r = \sqrt{x^2 + y^2}$

The term $|z|$ is also referred to as the absolute value of z.

The expression $|z_1 - z_2|$ represents the distance between the complex numbers z_1 and z_2.

Ex. Calculate the magnitude of $z = x + i\sqrt{1-2x}, x \in R$ and $x \leq \frac{1}{2}$

Sol. $|z| = \sqrt{x^2 + (1-2x)} = \sqrt{x^2 + 1 - 2x} = \sqrt{(x-1)^2} = |x-1|$

Argument of a Complex Number

The argument of a complex number is the angle formed by the line connecting the number, as represented in the Argand plane, with the origin in relation to the positive real axis.

If $z = x + iy$, then its argument is the solution to the equations $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$

The argument of z is alternatively referred to as the amplitude of z.

The principal argument is defined as the argument of any point within the interval $(-\pi, \pi]$.

Guidelines for determining the principal argument:

Let $z = x + iy$, x, y are non-zero real numbers and $\theta = \tan^{-1}\left|\frac{y}{x}\right|$. Then

(a) If $x > 0, y \geq 0$ (point lies in the first quadrant), then $\arg(z) = \theta$.

(b) If $x < 0, y \geq 0$ (point lies in the second quadrant), then $\arg(z) = \pi - \theta$.

(c) If $x < 0, y < 0$ (point lies in the third quadrant), then $\arg(z) = \theta - \pi$.

(d) If $x > 0, y < 0$ (point lies in the fourth quadrant), then $\arg(z) = -\theta$.

(e) The argument of $(0 + i0)$ is not defined.

e.g., For $k > 0$,

1. $\arg(k) = 0$
2. $\arg(-k) = \pi$
3. $\arg(ik) = \frac{\pi}{2}$
4. $\arg(-ik) = -\frac{\pi}{2}$

Ex. Determine the principal argument for all complex numbers represented by $\pm 1 \pm i$.

Sol Here four points exist. $(1+i), (-1+i), (-1-i), (1-i)$

$$\tan\theta = \left|\frac{y}{x}\right| = 1 \Rightarrow \theta = \frac{\pi}{4}$$

1. $\arg(1+i) = \theta = \frac{\pi}{4}$ (1^{st} quadrant)
2. $\arg(-1+i) = \pi - \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ (2^{nd} Quadrant)
3. $\arg(-1-i) = \theta - \pi = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$ (3^{rd} quadrant)
4. $\arg(1-i) = -\theta = -\frac{\pi}{4}$ (4^{th} quadrant)

Ex. If $z = x + iy$ determine the value of the argument of the modulus $|z|$.

Sol. Considering that $|z| \geq$ when $|z| > 0$, the argument of $|z|$ is 0. However, when $|z| = 0$, the argument of $|z|$ is not defined.

Ex. Evaluate the sum $\arg(1) + \arg(2i) + \arg(-i) + \arg(-1)$

Sol. We have, $\arg(1) = 0, \arg(-1) = \pi, \arg(2i) = \frac{\pi}{2}, \arg(-i) = -\frac{\pi}{2}$.

$$\text{Their sum } = 0 + \pi + \frac{\pi}{2} - \frac{\pi}{2} = \pi$$