ALGEBRA OF COMPLEX NUMBERS

Equality of Complex numbers

Two complex numbers a + ib and c + id are deemed equal if, and only if, a=c and b=d. In other words, their respective real and imaginary components must be identical.

If $a + ib = C_1$ and $c + id = C_2$

Then either $C_1 = C_2$ or $C_1 \neq C_2$

In the realm of imaginary numbers, the concept of order lacks definition since i is neither positive, zero, nor negative.

Therefore, statements like $C_1 > C_2$ or $C_2 > C_1$ are devoid of meaning unless both b and d are zero. In other words, $C_1 > C_2$ or $C_1 < C_2$ holds no significance unless both b and d are equal to zero.

Algebra of Complex Numbers

Addition of Two Complex Numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers.

Then, the sum z1 + z2 is defined as follows

$z_1 + z_2 = (a + c) + i\,(b + d)$, which is again a complex number

If $z_1 = 4 + i5$ and $z_0 = 3 + i(-4)$, then $z_1 + z_0 = (4 + 3) + i(5 - 4) = 7 + i$.

In the polar form or cis form

$$z_1 + z_2 = r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)$$

Properties of addition of complex numbers

The addition of complex numbers adheres to the following properties:

1. Closure Law:

The addition of two complex numbers results in another complex number. In other words, if z_1 and z_2 are any two complex numbers, then the sum, $z_1 + z_2$, is also a complex number.

Proof: Consider $z_1 = a + ib$ and $z_2 = c + id$ as two complex numbers. Then, the sum $z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d)$, which is also a complex number.

2. Commutative law:

For any pair of complex numbers, z_1 and z_2,

$$z_1 + z_2 = z_2 + z_1$$

Proof: Consider two complex numbers, $z_1 = a + ib$ and $z_2 = c + id$.

$$z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d)$$
$$z_2 + z_1 = (c + id) + (a + ib) = (c + a) + i(d + b)$$

However, it is known that the addition of real numbers is commutative, meaning a + c = c + a and b + d = d + b. Therefore, $z_1 + z_2$ equals $z_2 + z_1$.

3. Associative law:

For any triplet of complex numbers, z_1, z_2, and z_3

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_2)$$

Proof: Consider three complex numbers, $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, and $z_3 = a_3 + ib_3$. Then,

$$(z_1 + z_2) + z_3 = [(a_1 + ib_1) + (a_2 + ib_2)] + (a_3 + ib_3)$$
$$= [(a_1 + a_2) + i(b_1 + b_2)] + (a_3 + ib_3)$$
$$= [(a_1 + a_2) + a_3] + i[(b_1 + b_2) + b_3]$$
$$= [a_1 + (a_2 + a_3)] + i[b_1 + (b_2 + b_3)]$$
$$= (a_1 + ib_1) + [(a_2 + a_3) + i(b_2 + b_3)]$$
$$= (a_1 + ib_1) + [(a_2 + ib_2) + (a_3 + ib_3)]$$
$$= z_1 + (z_2 + z_3)$$

Thus, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

4. Additive identity:

For any complex number x + iy, there exists a complex number 0 + i0 such that the sum of (x + iy) and (0 + i0) is equal to (x + 0) + i(y + 0) = x + iy.

Here, 0 + i0 (denoted as 0), called the additive identity

5. Additive inverse:

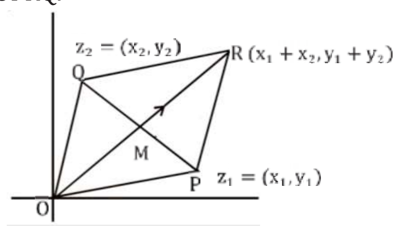
For any complex number z = a + ib, the complex number –a + i(–b) [denoted as –z] represents the additive inverse or negative of z.

Geometrical Representation of $z_1 + z_2$

Consider two given points P (z_1) and Q (z_2) on the Argand plane, as illustrated in the figure. Complete the parallelogram OPRQ, and let the diagonals intersect at point M. It is established that the diagonals of a parallelogram bisect each other. Therefore, M serves as the midpoint of the diagonal PQ, resulting in the ordered pair.

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ represents M, and consequently $R(x_1 + x_2, y_1 + y_2)$.

Hence, the addition of two complex numbers, denoted as (z1) and (z2), is depicted by the diagonal OR of the parallelogram OPRQ.



Example.$(2 + 6i) + (8 + 7i) = (2 + 8) + i(6 + 7) = 10 + 13i$

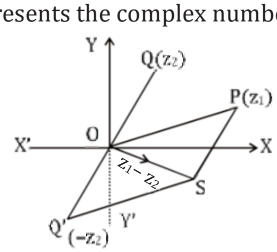
Subtraction

$$\begin{aligned} z_1 - z_2 &= (x_1 + iy_1) - (x_2 + iy_2) \\ &= (x_1 - x_2) + i(y_1 - y_2) \\ &= (\text{Re}(z_1) - \text{Re}(z_2)) + i(\text{lm}(z_1) - \text{Im}(z_2)) \end{aligned}$$

In polar form

$$\begin{aligned} z_1 - z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) - r_2(\cos\theta_2 + i\sin\theta_2) \\ &= (r_1\cos\theta_1 - r_2\cos\theta_2) + i(r_1\sin\theta_1 - r_2\sin\theta_2) \end{aligned}$$

In parallelogram OPSQ′, S represents the complex number $z_1 - z_2$.



Multiplication

We have $z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2$

$$z_1z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

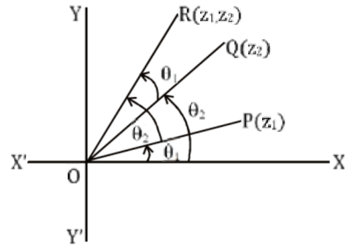
$$\text{Re}(z_1z_2) = \text{Re}(z_1)\text{Re}(z_2) - \text{Im}(z_1)\text{Im}(z_2)$$

$$\text{Im}(z_1z_2) = \text{Re}(z_1)\text{Im}(z_2) + \text{Re}(z_2)\text{Im}(z_1)$$

In Polar Form $z_1z_2 = r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2)$

$$= r_1r_2e^{i(\theta_1+\theta_2)}$$

$$z_1z_2 = r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$



Let $P(z_1 = r_1e^{i\theta_1})$ and $Q(z_2 = r_2e^{i\theta_2})$ be two given points in the Argand plane so that

$$OP = r_1 = |z_1|$$
$$OQ = r_2 = |z_2|$$
$$\angle XOP = \theta_1 = \arg(z_1)$$
$$\angle XOQ = \theta_2 = \arg(z_2)$$

Then, $z_1z_2 = r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)} = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

which shows that

$$|z_1z_2| = r_1r_2 = |z_1||z_2|$$
$$\arg(z_1z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$$

But principal $\text{amp}(z_1z_2) = \text{amp}(z_1) + \text{amp}(z_2) + 2k\pi$, where $k \in \{1, 0, -1\}$

In this observation, we note that the product of two complex numbers, z_1 and z_2, yields another complex number. This result can be achieved by rotating one complex number through the argument of the other in a counterclockwise direction around the origin of reference.

Properties of multiplication of complex numbers

1. Closure Law:

Multiplying two complex numbers results in another complex number. In other words, if z_1 and z_2 are two complex numbers, then their product, z_1z_2, is also a complex number.

Proof: Consider two complex numbers, where $z_1 = a + ib$ and $z_2 = c + id$. Now, the product z_1z_2 is obtained as $(a + ib)(c + id) = (ac – bd) + i(ad + bc)$, demonstrating that the result is indeed a complex number.

2. Commutative Law:

For any two complex numbers, z_1 and z_2, the product z1z2 is equal to z_2z_1.

Proof: Consider two complex numbers, where $z_1 = a + ib$ and $z_2 = c + id$.

$$z_1z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$$
$$z_2z_1 = (c + id)(a + ib) = (ca - db) + i(cb + da)$$

As the addition and multiplication of real numbers exhibit commutativity,

$$ac - bd = ca - db \text{ and } ad + bc = cb + da$$
$$z_1z_2 = z_2z_1$$

3. Associative Law:

For any three complex numbers, z_1, z_2, and z_3, the expression $(z_1z_2)z_3$ is equal to $z_1(z_2z_3)$.

Proof: Consider three complex numbers, where $z_1 = a + ib, z_2 = c + id$, and $z_3 = e + if$

$$(z_1z_2)z_3 = [(a + ib)(c + id)](e + if)$$
$$= [(ac - bd) + i(ad + bc)](e + if)$$
$$= [(ac - bd)e - (ad + bc)f] + i[(ac - bd)f + (ad + bc)e]$$
$$= [ace - bde - adf - bcf] + i[acf - bdf + ade + bce]$$
$$z_1(z_2z_3) = (a + ib)[(c + id)(e + if)]$$
$$= (a + ib)[(ce - df) + i(cf + de)]$$
$$= [a(ce - df) - b(cf + de)] + i[a(cf + de) + b(ce - df)]$$
$$= (ace - adf - bcf - bde) + i(acf + ade + bce - bdf)$$

As the addition of real numbers exhibits both commutative and associative properties,

$$ace - bde - adf - bcf = ace - adf - bcf - bde$$
$$acf - bdf + ade + bce = acf + ade + bce - bdf$$

Thus $$(z_1z_2)z_3 = z_1(z_2z_3)$$

4. Multiplicative identity

For any complex number x + iy, there exists a complex number 1 + i0 such that their product, (x + iy)(1 + i0), results in (x·1 – y·0) + i(x·0 + y·1).

$$= x + iy$$

1 + i0 is called the multiplicative identity.

5. Multiplicative inverse

For every non-zero complex number, expressed as z = x + iy (where x ≠ 0 and y ≠ 0), $\frac{x-iy}{x^2+y^2}$ such that $(x+iy)\frac{x-iy}{x^2+y^2} = 1 + i0 = 1$ (Multiplicative identity) and $\frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$ is referred to as the reciprocal or multiplicative inverse of x + iy.

The calculation for the multiplicative inverse (M.I) of a non-zero complex number is as follows:

$$x + iy = \frac{1}{x+iy} = \frac{1}{x+iy}\cdot\frac{x-iy}{x-iy} = \frac{x-iy}{x^2-i^2y^2} = \frac{x-iy}{x^2+y^2}$$
$$= \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$$

6. Distributive law

For any three complex numbers, z_1, z_2, z_3, we have

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3 \text{ and } (z_1 + z_2)z_3 = z_1z_3 + z_2z_3$$

Proof: Consider three complex numbers, where, $z_1 = a + ib$, $z_2 = c + id$, and $z_3 = e + if$.

$$z_1(z_2 + z_3) = (a + ib)[(c + id) + (e + if)]$$
$$= (a + ib)[(c + e) + i(d + f)]$$
$$= (a(c + e) - b(d + f)) + i(a(d + f) + b(c + e))]$$
$$= (ac + ae - bd - bf) + i(ad + af + bc + be)$$
$$z_1z_2 + z_1z_3 = [(a + ib)(c + id)] + [(a + ib)(e + if)]$$
$$= [(ac - bd) + i(ad + bc)] + [(ae - bf) + i(af + be)]$$
$$= (ac - bd + ae - bf) + i(ad + bc + af + be)$$

Since addition of real numbers is commutative and associative

$$ac + ae - bd - bf = ac - bd + ae - bf$$
$$\text{and } ad + af + bc + be = ad + bc + af + be$$

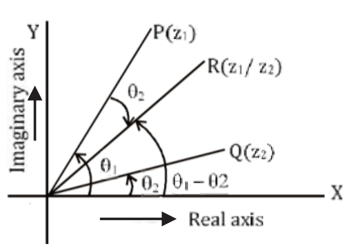
Thus $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

Similarly $(z_1 + z_2)z_3 = z_1z_3 + z_2z_3$

7. Division

$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} \times \frac{x_2-iy_2}{x_2-iy_2} = \frac{(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)}{x_2^2+y_2^2};\ (\text{Provided } z_2 \neq 0)$$
$$= \left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right) + i\left(\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}\right)$$

In polar form $\frac{z_1}{z_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)} = \frac{|z_1|}{|z_2|}e^{i(\arg(z_1)-\arg(z_2))}$



Which shows that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

But principal amp $\left(\frac{z_1}{z_2}\right) = \text{amp}(z_1) - \text{amp}(z_2) + 2k\pi$, where $k \in \{-1,0,1\}$

In this observation, it is noted that when two complex numbers are divided, the resultant number is also a complex number.

This can be achieved by rotating the complex number in the numerator through the amplitude of the complex number in the denominator in a clockwise direction around the origin of reference.

Square root of a complex number

Let $z = x + iy$ be a complex number then

$$\sqrt{z} = \sqrt{x + iy} = \pm(a + ib) \text{ (say)}$$

Squaring both sides

$$x + iy = a^2 - b^2 + i2ab.$$
$$a^2 - b^2 = x \text{ ... (i)}$$
$$\text{and } 2ab = y \text{ ... (ii)}$$
$$a^2 + b^2 = \sqrt{x^2 + y^2} \text{ ... (iii)}$$

From eq. (i) and (iii)

$$a^2 = \frac{\sqrt{x^2+y^2}+x}{2} \text{ and } b^2 = \frac{\sqrt{x^2+y^2}-x}{2}$$
$$\sqrt{x+iy} = \pm\left(\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} + i\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}\right)$$
$$\sqrt{x-iy} = \pm\left(\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} - i\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}\right)$$

Ex. If $z_1 = 3 - 2i, z_2 = 2 - i$ and $z_3 = 2 + 5i$, then find $z_1 + z_2 - 2z_3$.

Sol. $z_1 + z_2 - 2z_3 = (3 - 2i) + (2 - i) - 2(2 + 5i)$

$$= [(3 - 2i) + (2 - i)] - (4 + 10i)$$
$$= [(3 + 2) + i(-2 - 1)] - (4 + 10i)$$
$$= (5 - 3i) - (4 + 10i)$$
$$= (5 - 4) + i(-3 - 10)$$
$$= 1 - 13i$$

Ex. If $z_1 = 2 + 3i$ and $z_2 = -1 + 2i$, then find

(a) $z_1 + z_2$ (b) $z_1 - z_2$ (c) $z_1 \cdot z_2$ (d) $\frac{z_1}{z_2}$

Sol. We have, $z_1 = 2 + 3i$ and $z_2 = -1 + 2i$

(a) $z_1 + z_2 = (2 + 3i) + (-1 + 2i) = (2 - 1) + (3 + 2)i = 1 + 5i$

(b) $z_1 - z_2 = (2 + 3i) - (-1 + 2i) = (2 + 1) + (3 - 2)i = 3 + i$

(c) $z_1z_2 = (2 + 3i)(-1 + 2i) = (-2 - 6) + (4 - 3)i = -8 + i$

(d) $\frac{z_1}{z_2} = \frac{2+3i}{-1+2i} = (2 + 3i) \cdot \frac{1}{-1+2i}$

$$= (2 + 3i) \cdot \frac{1}{-1+2i} \cdot \frac{-1-2i}{-1-2i} = \frac{(2+3i)(-1-2i)}{(-1)^2-(2i)^2}$$
$$= \frac{1}{5}[(-2 + 6) + (-4 - 3)i] = \frac{1}{5}(4 - 7i)$$

Ex. If $z = 4 + 7i$ be a complex number, then find

1. Additive inverse of z

2. Multiplicative inverse of z

Sol. **1.** Additive inverse of $(4 + 7i) = (-4 - 7i)$

2. Multiplicative inverse of $(4 + 7i) = \frac{1}{4+7i} = \frac{1}{4+7i} \cdot \frac{4-7i}{4-7i}$

$$= \frac{(4-7i)}{(4)^2-(7i)^2} = \frac{4-7i}{65}$$
$$= \frac{4}{65} - \frac{7}{65}i$$