SIGN OF THE QUADRATIC EXPRESSION WITH GRAPHICAL PERSPECTIVE

It has been observed that $ax^2 + bx + c$ can be factored as $a(x - \alpha)(x - \beta)$, where α, and β are the roots of the corresponding equation $ax^2 + bx + c = 0$. This indicates that the sign of $ax^2 + bx + c$ is contingent on the value of 'a'.

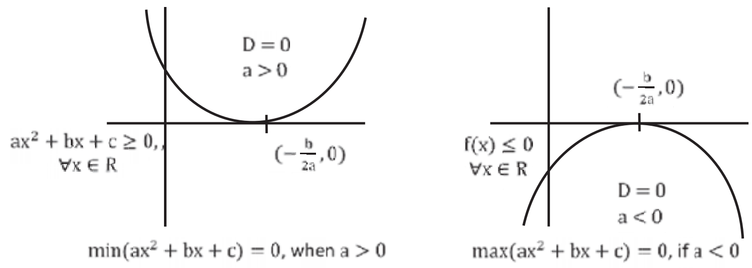
Case I: $D = 0$ (α and β are real and equal)

In this case $ax^2 + bx + c = a(x - \alpha)^2 \geq 0$, if $a > 0$

≤ 0, if $a < 0$

That is, the quadratic expression $ax^2 + bx + c$ is positive or negative depending on whether $a > 0$ or $a < 0$. The expression equals zero when $x = \alpha$.

Graphically, this is illustrated as follows:



Graph of $y = f(x)$ touches x-axis.

Case II: $D < 0$ ($b^2 - 4ac < 0$),

In this case the roots of the corresponding equation are conjugate complex numbers say $p + iq$ and $p - iq$.

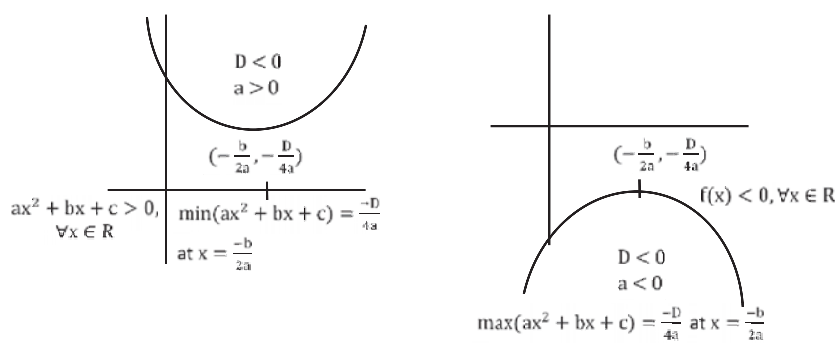
Let $\alpha = p + iq$, $\beta = p - iq$.

Now, $ax^2 + bx + c = a(x - \alpha)(x - \beta)$

$= a[(x - p) - iq][(x - p) + iq]$

$= a[(x - p)^2 + q^2] >< 0$ according as $a >< 0$; because $(x - p)^2 + q^2 > 0$ for real values of x, p, q

This indicates that the expression $ax^2 + bx + c$ is consistently positive or negative depending on whether $a > 0$ or $a < 0$, respectively. This graphical representation is illustrated below.



Graph of $y = f(x)$ neither touches nor intersects x-axis.

Case III: > 0 i.e., ($b^2 - 4ac > 0$),

In this case roots are real and distinct,

i.e., $\alpha, \beta \in R$ and $\alpha \neq \beta$.

Let$\alpha < \beta$

For $x < \alpha \Rightarrow x < \beta$

$x - \alpha < 0$ and $x - \beta < 0$

$(x - \alpha)(x - \beta) > 0$

$ax^2 + bx + c = a(x - \alpha)$

$(x - \beta) \gtrless 0$

If $a \gtrless 0$

For $\alpha < x < \beta$

$x - \alpha > 0$ and $x - \beta < 0$

$(x - \alpha)(x - \beta) < 0$

$ax^2 + bx + c = a(x - \alpha)$

$(x - \beta) \gtrless 0$

According as $a \gtrless 0$

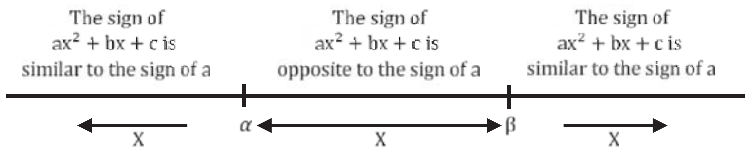
For $x > \beta \Rightarrow x > \alpha$

$(x - \alpha)(x - \beta) > 0$

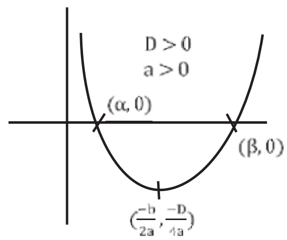
$ax^2 + bx + c \gtrless 0$

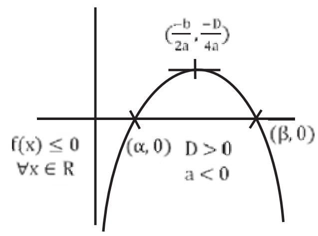
According as $a \gtrless 0$

The above cases can be interpreted on the number line as.



The intersection of the graph $y = f(x)$ with the x-axis is evident from the graphs below, revealing two points of intersection.





$\min(ax^2 + bx + c) = \frac{-D}{4a}$ which occurs at $x = \frac{-b}{2a}$

$ax^2 + bx + c \geq 0, \forall x \in (-\infty, \alpha] \cup [\beta, \infty)$

and $ax^2 + bx + c \leq 0, \forall x \in [\alpha, \beta]$

Graph of $y = f(x)$ intersects x-axis at two distinct points.

$\max(ax^2 + bx + c) = \frac{-D}{1a}$ which occurs at $x = \frac{-b}{2a}$

$ax^2 + bx + c \leq 0, \forall x \in (-\infty, \alpha] \cup [\beta, \infty)$

and $ax^2 + bx + c \geq 0, \forall x \in [\alpha, \beta]$