RELATION BETWEEN ROOTS AND COEFFICIENTS OF A POLYNOMIAL EQUATION

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots \ldots . . + a_{n-1}x + a_n = 0$$

Let $\alpha_1, \alpha_2, \alpha_3, \ldots \ldots . \alpha_{n-1}, \alpha_n$ be the roots of the given equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots \ldots . . + a_{n-1}x + a_n = 0, a_0 \neq 0$$

Then $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots \ldots . . + a_{n-1}x + a_n$

$$\equiv a_0(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots \ldots . . (x - \alpha_n) = 0$$

$$\equiv a_0[x^n - (\alpha_1 + \alpha_2 + \cdots \ldots + \alpha_n)x^{n-1} + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \cdots \ldots + \alpha_1\alpha_n + \alpha_2\alpha_3 + \cdots \ldots . + \alpha_{n-1}\alpha_n)x^{n-2}$$
$$-(\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \cdots \ldots \ldots + \alpha_{n-2}\alpha_{n-1}\alpha_n)x^{n-3} + \cdots \ldots . . + (-1)^n\alpha_1\alpha_2\alpha_3\alpha_4 \ldots \ldots . . \alpha_n] = 0.$$

$$\Rightarrow \frac{a_0}{a_0} = \frac{-a_0(\alpha_1 + \alpha_2 + \cdots \ldots \ldots + \alpha_n)}{a_1} = \frac{a_0(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots \ldots \ldots + \alpha_{n-1}\alpha_n)}{a_2}$$

$$= \frac{-a_0(\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \cdots \ldots . . + \alpha_{n-2}\alpha_{n-1}\alpha_n)}{a_3} = \cdots \ldots . . = \frac{(-1)^n a_0\alpha_1\alpha_2\alpha_3 \ldots \ldots . . \alpha_n}{a_n}$$

$$\sum_{i=1}^{n} \alpha_i = \alpha_1 + \alpha_2 + \alpha_3 + \cdots \ldots . . + \alpha_n = \text{ sum of all roots } = (-1)^1 \frac{a_1}{a_0} = (-1)^1 \frac{\text{coeff. of } x^{n-1}}{\text{coeff. of } x^n}$$

$$\sum_{1 \le i < j \le n} \alpha_i\alpha_j = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots \ldots . . + \alpha_{n-1}\alpha_n = \text{ sum of product of roots taken two at a time}$$

$$(-1)^2 \frac{a_2}{a_0} = (-1)^2 \frac{\text{coeff. of } x^{n-2}}{\text{coeff. of } x^n}$$

$$\sum_{1 = i < j < k \le n} \alpha_i\alpha_j\alpha_k = \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \cdots \ldots +$$

$$\alpha_{n-2}\alpha_{n-1}\alpha_n = \text{ sum of product of roots taken three at a time}$$

$$(-1)^3 \frac{a_3}{a_0} = (-1)^3 \frac{\text{coeff. of } x^{n-3}}{\text{coeff. of } x^n}$$

$$\text{Product of all roots } = \alpha_1\alpha_2\alpha_3 \ldots \ldots \alpha_n = (-1)^n \frac{a_n}{a_0} = (-1)^n \frac{\text{constant term}}{\text{coeff. of } x^n}$$

Ex. If the roots of the cubic equation $x^3 - 9x^2 + a = 0$ form an arithmetic progression (A.P.), determine one of the roots and the value of 'a'.

Sol. Let $\alpha - \beta, \alpha, \alpha + \beta$ be roots of the given equation $x^3 - 9x^2 + a = 0$, then

$$\alpha - \beta + \alpha + \alpha + \beta = 9$$
$$\alpha = 3$$

One of roots of the given equation is 3 substituting $x = 3$, we get

$$27 - 81 + a = 0$$
$$a = 54$$