QUADRATIC EQUATION

A mathematical expression in the format

$$ax^2 + bx + c = 0, a \neq 0, b, c \in R,$$

It is termed a quadratic equation, wherein a, b, and c represent the coefficients of x^2, x, and the constant term, respectively. a root of the equation is a numerical value (real or complex) α that fulfills the equation.

$$a\alpha^2 + b\alpha + c = 0.$$

The roots of $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} = \frac{-b}{2a} + \frac{\sqrt{D}}{2a}, \frac{-b}{2a} - \frac{\sqrt{D}}{2a}, \text{where } D = b^2 - 4ac.$$

The distinction between the roots of the quadratic equation $ax^2 + bx + c = 0$ can be determined by the expression $D = b^2 - 4ac$, termed as the discriminant of the equation. If α and β are the roots of the specified quadratic equation $ax^2 + bx + c = 0$, then

$$\alpha + \beta = \frac{-b}{a} = \text{ Sum of roots } = -\frac{\text{coefficient of x}}{\text{coefficient of } x^2}$$

$$\text{and } \alpha\beta = \frac{c}{a} = \text{ Product of roots } = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Other expressions involving α and β can be determined as follows:

1. $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
2. $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$
3. $\alpha^3 + \beta^3 = (\alpha + \beta)((\alpha + \beta)^2 - 3\alpha\beta)$
4. $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$
   $= ((\alpha + \beta)^2 - 2\alpha\beta)^2 - 2\alpha^2\beta^2$
5. $\alpha^3 - \beta^3 = (\alpha - \beta)(\alpha^2 + \beta^2 + \alpha\beta)$
   $= (\pm\sqrt{(\alpha + \beta)^2 - 4\alpha\beta})((\alpha + \beta)^2 - \alpha\beta)$

Newton's Theorem

If α and β represent the roots of the quadratic equation $ax^2 + bx + c = 0, a \neq 0$ and $S_n = \alpha^n + \beta^n$, $n \in N$ then $aS_{n+2} + bS_{n+1} + cS_n = 0$

Note. Newton's theorem is applicable to a polynomial equation of any degree.

Ex. If α and β are the roots of the equation $ax^2 + bx + c = 0$, then calculate $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}, \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ and $\frac{\alpha^3}{\beta} + \frac{\beta^3}{\alpha}$.

Sol. Thus, α, and β serve as roots for the equation $ax^2 + bx + c = 0$.

$$\alpha + \beta = \frac{-b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta} = \frac{(\frac{-b}{a})^2-2\frac{c}{a}}{c/a} = \frac{b^2-2ac}{ac}$$

$$\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3+\beta^3}{\alpha\beta} = \frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta} = \frac{(\frac{-b}{a})^3-3\frac{c}{a}(\frac{-b}{a})}{c/a} = \frac{-b^3+3abc}{a^2c}$$

$$\text{Moreover, } \frac{\alpha^3}{\beta} + \frac{\beta^3}{\alpha} = \frac{\alpha^4+\beta^4}{\alpha\beta} = \frac{(\alpha^2+\beta^2)^2-2\alpha^2\beta^2}{\alpha\beta}$$

$$= \frac{[(\alpha+\beta)^2-2\alpha\beta]^2-2\alpha^2\beta^2}{\alpha\beta}$$

$$= \frac{[(\frac{-b}{a})^2-\frac{2c}{a}]^2-\frac{2c^2}{a^2}}{c/a}$$

$$= \frac{(b^2-2ac)^2-2a^2c^2}{a^3c}$$

$$= \frac{b^4-4ab^2c+2a^2c^2}{a^3c}$$

Ex. If α is a root of the equation $4x^2 + 2x - 1 = 0$, demonstrate that the other root of the equation can be expressed as $4\alpha^3 - 3\alpha$.

Sol. Consider β as the second root of the provided equation $4x^2 + 2x - 1 = 0$, where α is one of the roots.

Then sum of roots $= \alpha + \beta = \frac{-2}{4} = -\frac{1}{2}$

$$\Rightarrow \beta = -\frac{1}{2} - \alpha$$

$$\because \alpha \text{ is a root of } 4x^2 + 2x - 1 = 0$$

$$\Rightarrow 4\alpha^2 + 2\alpha - 1 = 0$$

$$\Rightarrow 4\alpha^2 = 1 - 2\alpha; 2\alpha^2 = \frac{1}{2} - \alpha$$

$$\Rightarrow 4\alpha^3 = \alpha - 2\alpha^2$$

$$\Rightarrow 4\alpha^3 - 3\alpha = \alpha - 2\alpha^2 - 3\alpha$$

$$= -2\alpha - \frac{1}{2} + \alpha (\because 2\alpha^2 = \frac{1}{2} - \alpha)$$

$$= -\frac{1}{2} - \alpha = \beta$$

Ex. Given that α and β are the solutions of the equation.

$$x^2 - 7x + 17 = 0 \text{ and } a_n = \alpha^n + \beta^n \text{ for } n \geq 1, \text{then}$$

find the value of $\frac{a_{10}+17a_8}{a_0}$.

Sol. By Newton's theorem

$$a_{10} - 7a_9 + 17a_8 = 0$$

$$\frac{a_{10} + 17a_8}{a_9} = 7$$

Nature of Roots of Quadratic Equation $ax^2 + bx + c = 0, a \neq 0, a, b, c$ all are real numbers.

1. The solutions of $ax^2 + bx + c = 0$ are real and equal if and only if the discriminant $D = b^2 - 4ac$ is equal to 0.
2. When $D > 0$, the roots are real and distinct.
3. If D is a perfect square, the roots of $ax^2 + bx + c = 0$ are rational, provided that a, b, and c are rational. Otherwise, the roots are irrational.
4. If $a = 1$ and b, c are integers, and the roots are rational, then the roots of the quadratic equation must be integers.
5. In case $\alpha + \beta$ represents an irrational root of the quadratic equation, $a - \sqrt{\beta}$ also serves as a root for the quadratic equation, given that all coefficients a, b, c are rational.
6. If $D < 0$, the roots are imaginary numbers. If $\alpha + i\beta$ is one of the roots in the given equation, with real coefficients and $\beta \neq 0$, then the other root must be its conjugate $\alpha - i\beta$, and vice versa, where $\alpha, \beta \in R$, and $i = \sqrt{-1}$. In simpler terms, we can state that complex roots of an equation (if they exist) always appear in conjugate pairs, provided that coefficients a, b, c are real numbers.
7. If the roots of the given quadratic equation $ax^2 + bx + c = 0$ have equal magnitudes but opposite signs, then $b = 0$, meaning the coefficient of x is zero.

   If $\alpha, -\alpha$ are roots of $ax^2 + bx + c = 0$, then

   sum of roots $= -\frac{b}{a} = \alpha + (-\alpha) = 0 \quad \Rightarrow b = 0.$

   Observation: The roots of $x^2 - 4 = 0$ are $x = 2, -2$ and that of $x^2 + 9 = 0$ are $3i$ and $-3i$.
8. If the roots of the equation $ax^2 + bx + c = 0$ are reciprocals, then the coefficient of x^2 is equal to the constant term.

   i.e., $a = c$. (In this case product of roots $= 1 = \alpha \cdot \frac{1}{\alpha} = \frac{c}{a}$)

   Observation: The roots of $2x^2 + 5x + 2 = 0$ are -2 and $-\frac{1}{2}$ which are reciprocal as coefficient of $x^2 =$ constant term $= 2$

9. If the sum of the coefficients of $ax^2 + bx + c = 0$ is zero (i.e., if $a + b + c = 0$), then the roots of the equation are 1 and. $\frac{c}{a}$

Similarly if $a - b + c = 0$, then roots of $ax^2 + bx + c = 0$ are -1 and $-\frac{c}{a}$.

10. If the quadratic equation is fulfilled by more than two roots (whether real or complex), it transforms into an identity, and in such instances, $a = b = c = 0$.

Proof:

Let x_1, x_2, x_3 be three distinct roots of $ax^2 + bx + c = 0$ then

$ax_1^2 + bx_1 + c = 0$ ...(i)

$ax_2^2 + bx_2 + c = 0$ ...(ii)

$ax_3^2 + bx_3 + c = 0$ ...(iii)

By (i) - (ii), $a(x_1 + x_2) + b = 0$ ...(iv) $(x_1 \neq x_2)$

By (ii) - (iii), $a(x_2 + x_3) + b = 0$ ...(v) $(x_2 \neq x_3)$

Now by (iv), (v), $a = 0, b = 0$ $(x_1 \neq x_3)$

In equation (i) putting $a = b = 0$, we get $c = 0$

11. A quadratic equation with roots α and β can be expressed as $(x - \alpha)(x - \beta) = 0$, which is equivalent to $x^2 - (\alpha + \beta)x + \alpha\beta = 0$, i.e., $ax^2 + bx + c \equiv a(x - \alpha)(x - \beta)$.

Ex. Examine the characteristics of the roots of the equation x^2 – 2008x + 2007 = 0.

Sol. Here, $D = (2008)^2 - 4.2007 = (2007 + 1)^2 - 4.1.2007$

$= (2007 - 1)^2 = (2006)^2 =$ a perfect square

Roots are $\frac{2008+2006}{2}, \frac{2008-2006}{2}$

i.e. 2007 and 1 are roots of the qiven equation.

Second solution

We note that the sum of the coefficients of the equation equals $1 - 2008 + 2007 = 0$. Therefore, the roots of the given equation are 1 and 2007.

Ex. If $2 + i\sqrt{3}$ be is a root of the equation $x^2 + px + q = 0$, where p and q are real, determine the values of p and q.

Sol. Therefore, if $2 + i\sqrt{3}$ is a root of the equation, $x^2 + px + q = 0$, the other root must be $2 - i\sqrt{3}$

Now Sum of roots $= -p = (2 + i\sqrt{3}) + (2 - i\sqrt{3}) = 4$

$\Rightarrow p = -4$.

Product of roots $= q = (2 + i\sqrt{3})(2 - i\sqrt{3}) = 4 - 3i^2 = 7$

Ex. If α and β represent the roots of the equation $(x - a)(x - b) = c$, where $c \neq 0$, find the roots of the equation $(x - \alpha)(x - \beta) + c = 0$.