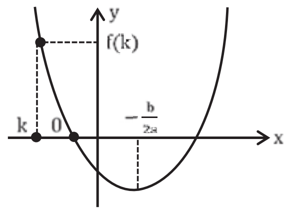
# LOCATION OF ROOTS

Consider α and β as real solutions of the equation $f(x) = ax^2 + bx + c = 0$, where k is a real constant.

**Case I:** k is smaller than both roots of the equation $f(x) = 0$.

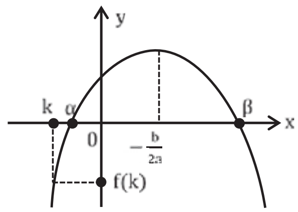
**(a)** When $a > 0$

We see from graph $f(k) > 0, D \geq 0$ and $k < -\frac{b}{2a}$ ... (1)




**(b)** When $a < 0$

We can see from graph $f(k) < 0, D \geq 0$ and $k < -\frac{b}{2a}$ ... (2)



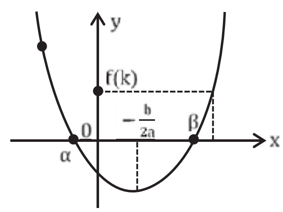

From (1) and (2), we conclude that when k is less than both the roots of $f(x) = 0$ we must have $af(k) > 0$;

$$D \geq 0 \text{ and } k < -\frac{b}{2a}$$

**Case II:** k exceeds both roots of the equation $f(x) = 0$.

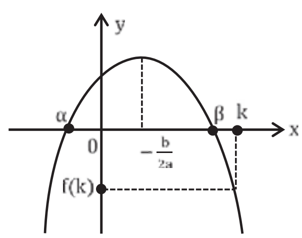
**(a)** When $a > 0$

We can see from graph $f(k) > 0, D \geq 0, k > -\frac{b}{2a}$ ... (1)




**(b)** When $a < 0$

We can see from graph $f(k) < 0, D \geq 0, k > -\frac{b}{2a}$ ... (2)




**Case III:** The value of k falls within the range of both roots of the equation $f(x) = 0$.

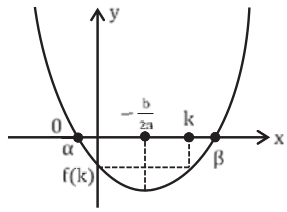
**(a)** When $a > 0$

We can see from graph $f(k) < 0$ and $D > 0$

But k can be smaller, greater or equal to $-\frac{b}{2a}$

i.e., no conclusion can be made for $-\frac{b}{2a}$

Only two conditions are there $f(k) < 0$ and $D > 0$ ... (1)

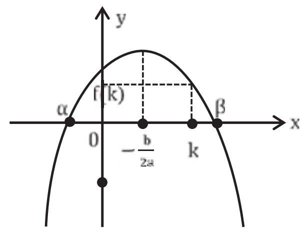



**(b)** When $a < 0$

We can see from graph $f(k) > 0$ and $D > 0$

But k can be smaller, greater or equal to $-\frac{b}{2a}$

i.e., no conclusion can be made for $-\frac{b}{2a}$ … (2)




Only two conditions are there $f(k) > 0$ and $D > 0$ …(2)

From (1) and (2) we conclude that when k lies between the roots of $f(x) = 0$ we must have $af(k) < 0$ and $D > 0$

**Ex.** Identify the values of m in the set of real numbers for which the equation $2x^2 - 2(2m + 1)x + m(m + 1) = 0$ holds.

**1.** Are both roots less than 2?

**2.** Are both roots greater than 2?

**3.** Do both roots fall within the interval (2, 3)?

**4.** Does exactly one root lie within the interval (2, 3)?

**5.** Is one root less than 1, while the other is greater than 1?

**6.** Is one root greater than 3 and the other smaller than 2?

**7.** Are the roots α and β such that both 2 and 3 lie between them?

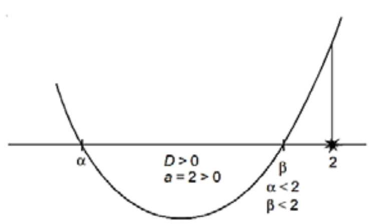
**Sol.** Let $f(x) = 2x^2 - 2(2m+1)x + m(m+1), m \in R$

Here
$$D = 4(2m+1)^2 - 4 \cdot 2m(m+1)$$
$$= 4m^2 + 4m - 2m^2 - 2m + 1$$
$$= 2m^2 + 2m + 1$$
$$= (m+1)^2 + m^2 > 0$$
$$D > 0$$

D is always positive for all real values of m.

Here coefficient of $x^2 = 2 > 0$.

**1.** When both roots of $f(x) = 0$ are smaller than 2, then clearly $f(2) > 0$ and $\frac{-b}{2a} = \frac{2(2m+1)}{2\times 2} < 2$

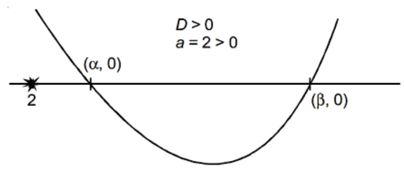



$$\Rightarrow 8 - 4(2m+1) + m^2 + m > 0 \text{ and } m < \frac{3}{2}$$
$$\Rightarrow m^2 - 7m + 4 > 0 \text{ and } m < \frac{3}{2}$$
$$\Rightarrow m \in \left(-\infty, \frac{7-\sqrt{33}}{2}\right) \cup \left(\frac{7+\sqrt{33}}{2}, \infty\right) \text{ and } m < \frac{3}{2}$$


But $\frac{7+\sqrt{33}}{2} > \frac{3}{2}$

Required interval is $m \in (-\infty, \frac{7-\sqrt{33}}{2})$.

**2.** If both roots of $f(x) = 0$ are greater than 2, it is evident that $f(2) > 0$, and $\frac{-b}{2a} = \frac{2(2m+1)}{2\times2} > 2$

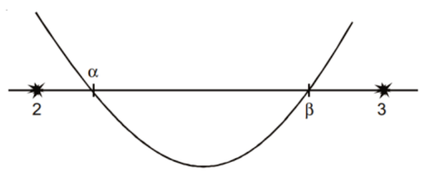



$$\Rightarrow m \in (-\infty, \frac{7-\sqrt{33}}{2}) \cup (\frac{7+\sqrt{33}}{2}, \infty) \text{ and } m > \frac{3}{2}$$

$$\Rightarrow m \in (\frac{7+\sqrt{33}}{2}, \infty)$$

**3.** For both roots lying in the interval (2, 3), it is necessary that $f(2) > 0, f(3) > 0$, and

$$2 < \frac{-b}{2a} < 3.$$




$$\Rightarrow m^2 - 7m + 4 > 0 \text{ and } m^2 - 11m + 12 > 0 \text{ and } 2 < \frac{2m+1}{2} < 3$$

$$\Rightarrow m \in (-\infty, \frac{7-\sqrt{33}}{2}) \cup (\frac{7+\sqrt{33}}{2}, \infty), m \in (-\infty, \frac{11-\sqrt{73}}{2}) \cup (\frac{11+\sqrt{73}}{2}, \infty) \text{ and } \frac{3}{2} < m < \frac{5}{2}$$

There is no value of m that satisfies the aforementioned conditions.

**4.** For exactly one root to lie in the interval (2, 3), it is necessary that $f(2)\, f(3) < 0$.

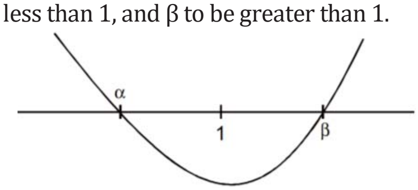
$$\Rightarrow \text{ either } f(2) < 0 \text{ and } f(3) > 0 \text{ or } f(2) > 0 \text{ and } f(3) < 0.$$

$$\Rightarrow m \in (\frac{7-\sqrt{33}}{2}, \frac{7+\sqrt{33}}{2}) \text{ and } m \in (-\infty, \frac{11-\sqrt{73}}{2}) \cup (\frac{11+\sqrt{73}}{2}, \infty)$$

$$m \in (-\infty, \frac{7-\sqrt{33}}{2}) \cup (\frac{7+\sqrt{33}}{2}, \infty) \text{ and } m \in (\frac{11-\sqrt{73}}{2}, \frac{11+\sqrt{73}}{2})$$

$$m \in (\frac{7-\sqrt{33}}{2}, \frac{11-\sqrt{73}}{2}) \cup (\frac{7+\sqrt{33}}{2}, \frac{11+\sqrt{73}}{2})$$

**5.** Consider α to be less than 1, and β to be greater than 1.




$$f(1) < 0$$

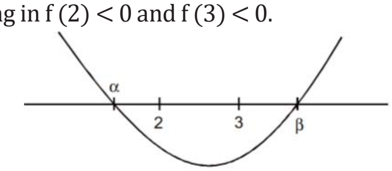
$$2 - 2(2m+1) + m(m+1) < 0$$

$$m^2 - 3m < 0$$

$$m(m-3) < 0$$

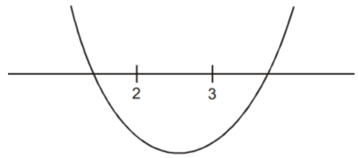
$$m \in (0,3).$$

**6.** Suppose α is less than 2 and β is greater than 3. In such a scenario, both 2 and 3 fall between the roots, resulting in $f(2) < 0$ and $f(3) < 0$.



$$m^2 - 7m + 4 < 0 \text{ and } m^2 - 11m + 12 < 0$$
$$m \in \left(\frac{7-\sqrt{33}}{2}, \frac{7+\sqrt{33}}{2}\right) \text{ and } m \in \left(\frac{11-\sqrt{73}}{2}, \frac{11+\sqrt{73}}{2}\right)$$
$$m \in \left(\frac{11-\sqrt{73}}{2}, \frac{7+\sqrt{33}}{2}\right)$$

**7.** Since both 2 and 3 are situated between $\alpha$ and $\beta$, it follows that $f(2) < 0$ and $f(3) < 0$.




$$m \in \left(\frac{11-\sqrt{73}}{2}, \frac{7+\sqrt{33}}{2}\right)$$