CONDITION FOR COMMON ROOT(S) OF QUADRATIC EQUATIONS

1. Condition for One Common Root between two Quadratic Equations

Let α be a shared root between the two given quadratic equations.

$$ax^2 + bx + c = 0 \text{ and } a_1x^2 + b_1x + c_1 = 0$$

Then $a\alpha^2 + b\alpha + c = 0$

And $a_1\alpha^2 + b_1\alpha + c_1 = 0$

Through cross-multiplication, we obtain:

$$\frac{\alpha^2}{bc_1 - b_1c} = \frac{\alpha}{a_1c - ac_1} = \frac{1}{ab_1 - a_1b}$$

$$\alpha = \frac{bc_1 - b_1c}{a_1c - ac_1} = \frac{a_1c - ac_1}{ab_1 - a_1b}$$

$$(a_1c - ac_1)^2 = (ab_1 - a_1b)(bc_1 - b_1c)$$

which is the required condition.

Ex. If the quadratic equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ (where $a \neq b$) share a common root, determine the numerical value of a + b.

Sol. Consider α as a shared root between the given quadratic equations.

$$\alpha^2 + a\alpha + b = 0$$

$$\alpha^2 + b\alpha + a = 0$$

On subtraction $(a - b)\alpha + b - a = 0 \Rightarrow \alpha = 1.$

The common root is 1 . Putting $\alpha = 1$, we get

$$1 + a + b = 0 \Rightarrow a + b = -1 \Rightarrow |a + b| = 1$$

Absolute value of $(a + b) = 1$

Numerical value of a + b is -1 .

2. Condition for two Quadratic Equations to have both Roots Common

Let α, β be roots of given quadratic equations $ax^2 + bx + c = 0$ and $a_1x^2 + b_1x + c_1 = 0$

$$\left.\begin{aligned} \alpha + \beta &= \frac{-b}{a} = \frac{-b_1}{a_1} \\ \alpha\,\alpha\beta &= \frac{c}{a} = \frac{c_1}{a_1} \end{aligned}\right\} \begin{aligned} &\Rightarrow \frac{a}{a_1} = \frac{b}{b_1} \\ &\Rightarrow \frac{a}{a_1} = \frac{c}{c_1} \end{aligned}$$

$$\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$$

Which is the required condition

Ex. If a, b, c ∈ R, determine a relationship between a, b, and c such that the two quadratic equations $ax^2 + bx + c = 0$ and $1003x^2 + 1505x + 2007 = 0$ share a common root.

Sol. We note that the discriminant of the quadratic equation

$$1003x^2 + 1505x + 2007 = 0 \text{ is } (1505)^2 - 4 \times 1003 \times 2007 < 0.$$

Roots of the equation are complex.

We are aware that complex roots of an equation appear in conjugate pairs. In this scenario, whenever α is a common root between the given equations, the other root automatically becomes common.

Thus, the given equations share both roots, and consequently,

$$\frac{a}{1003} = \frac{b}{1505} = \frac{c}{2007} = \frac{a+c}{3010}$$

$$2b = a + c$$

a, b, c are in A.P

which is the required relation.

Ex. If the equation $x^2 + 3x + 9 = 0$ and $ax^2 + bx + c = 0$ share a common root, and $a, b, c \in N$ (natural numbers), then

1. Determine a relation between a, b, c, and
2. Find the minimum value of $a + b + c$.

Sol. The roots of the equation $x^2 + 3x + 9 = 0$ are evidently complex, given that its discriminant is less than 0. Therefore, the coefficients of like terms in the given equations are proportional.

$$\frac{a}{1} = \frac{b}{3} = \frac{c}{9} = k, \text{ where } k \geq 1 \text{ as } a, b, c \in N.$$

Here we observe that $b^2 = (3k)^2 = 9k^2 = k(9k) = ac$

a, b, c are in G.P.

Now $a + b + c = k + 3k + 9k = 13k, k \geq 1$, a natural number

Minimum value of $a + b + c = 13$