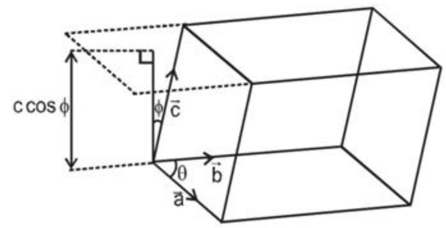
SCALAR AND VECTOR TRIPLE PRODUCTS AND THEIR GEOMETRICAL INTERPRETATION

Scalar Triple Product / Box Product / Mixed Product :

(A) The scalar triple product of three vectors $\vec{a}, \vec{b} \& \vec{c}$ is defined as : $(\vec{a} \times \vec{b}) \cdot \vec{c} = |\vec{a} \| \vec{b} \| \vec{c}|\sin\theta\cos\phi$ where θ is the angle between $\vec{a} \& \vec{b} \& \phi$ is the angle between $\vec{a} \times \vec{b} \& \vec{c}$. It is also defined as $[\vec{a} \quad \vec{b} \quad \vec{c}]$, spelled as box product.



(B) In a scalar triple product the position of dot & cross can be interchanged i.e. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ OR $[\vec{a} \quad \vec{b} \quad \vec{c}] = [\vec{b} \quad \vec{c} \quad \vec{a}] = [\vec{c} \quad \vec{a} \quad \vec{b}]$

(C) $\vec{a} \cdot (\vec{b} \times \vec{c}) = -a \cdot (\vec{c} \times \vec{b})$ i.e. $[\vec{a}\vec{b}\vec{c}] = -[\vec{a}\vec{c}\vec{b}]$

(D) If $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow [\vec{a} \quad \vec{b} \quad \vec{c}] = 0 \Rightarrow \vec{a}, \vec{b}, \vec{c}$ are linearly dependent.

(E) Scalar product of three vectors, two of which are equal or parallel is 0 i.e. $[\vec{a}\vec{b}\vec{c}] = 0$

(F) $[ijk] = 1$; $[K\vec{a}\vec{b}\vec{c}] = K[\vec{a}\vec{b}\vec{c}]$; $[(\vec{a} + \vec{b})\vec{c}\vec{d}] = [\vec{a}\vec{c}\vec{d}] + [\vec{b}\vec{c}\vec{d}]$

(G) (i) The Volume of the tetrahedron OABC with O as origin & the pv's of A, B and C being $\vec{a}, \vec{b} \& \vec{c}$ are given by $V = \frac{1}{6}[\vec{a} \quad \vec{b} \quad \vec{c}]$

(ii) volume of the parallelopiped whose three coterminous edges are represented by $\vec{a}, \vec{b} \& \vec{c}$ is $[\vec{a}\vec{b}\vec{c}]$.

(ii) Remember that :

(i) $[\vec{a} - \vec{b} \quad \vec{b} - \vec{c} \quad \vec{c} - \vec{a}] = 0$

(ii) $[\vec{a} + \vec{b} \quad \vec{b} + \vec{c} \quad \vec{c} + \vec{a}] = 2[\vec{a} \quad \vec{b} \quad \vec{c}]$

(iii) $|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a} \times \vec{b}\vec{b} \times \vec{c}\vec{c} \times \vec{a}| = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} & \vec{a} \cdot \vec{c} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{b} & \vec{b} \cdot \vec{c} \\ \vec{c} \cdot \vec{a} & \vec{c} \cdot \vec{b} & \vec{c} . \vec{c} \end{vmatrix}$

Vector Triple Product

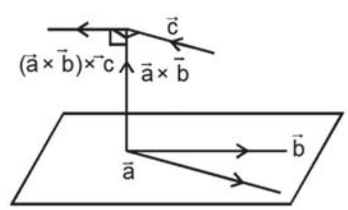
Let $\vec{a}, \vec{b} \& \vec{c}$ be any three vectors,

then the expression $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector & is called a vector triple product.

Geometrical interpretation of $\vec{a} \times (\vec{b} \times \vec{c})$

Consider the expression $\vec{a} \times (\vec{b} \times \vec{c})$ which itself is a vector, since it is a cross product of two vectors $\vec{a} \& (\vec{b} \times \vec{c})$.

Now $\vec{a} \times (\vec{b} \times \vec{c})$ is vector perpendicular to the plane containing $\vec{a} \& (\vec{b} \times \vec{c})$ but $(\vec{b} \times \vec{c})$ is a vector perpendicular to the plane $\vec{b} \& \vec{c}$, therefore $\vec{a} \times (\vec{b} \times \vec{c})$ is vector lies in the plane of and perpendicular to .



Hence we can express $\vec{a} \times (\vec{b} \times \vec{c})$ in terms of $\vec{b} \& \vec{c}$ i.e. $\vec{a} \times (\vec{b} \times \vec{c}) = x\vec{b} + y\vec{c}$ where x & y are scalars.

(A) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

(B) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$

(C) $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

Scalar Product of Four Vectors

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$$

$$= \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

Vector Product of Four Vectors

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\vec{b}\vec{d}]\vec{c} - [\vec{a}\vec{b}\vec{c}]\vec{d}$$

$$= [\vec{a}\vec{c}\vec{d}]\vec{b} - [\vec{b}\vec{c}\vec{d}]\vec{a}$$