RECIPROCAL SYSTEM OF VECTORS

If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors such that $[\vec{a}, \vec{b}, \vec{c}] \neq 0$, then three vectors defined by

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}, \vec{b}, \vec{c}]}, \vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}, \vec{b}, \vec{c}]}, \vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}, \vec{b}, \vec{c}]}$$

are called reciprocal system of vectors to the vectors $\vec{a}, \vec{b}, \vec{c}$ with the following properties

(i) $\vec{a} \cdot \vec{a}' = \vec{b} \cdot \vec{b}' = \vec{c} \cdot \vec{c}' = 1$

(ii) $\vec{a} \cdot \vec{b}' = \vec{a} \cdot \vec{c}' = \vec{b} \cdot \vec{c}' = \vec{b} \cdot \vec{a}' = \vec{c} \cdot \vec{a}' = \vec{c} \cdot \vec{b}' = 0$

(iii) $\left[\vec{a}\vec{b}\vec{c}\right] = \frac{1}{\left[\vec{a}'\vec{b}'\vec{c}'\right]}$

(iv) $\vec{a} \cdot \vec{a}' + \vec{b} \cdot \vec{b}' + \vec{c} \cdot \vec{c}' = 3$

SOLVING VECTOR EQUATION

Sometimes one is given a vector equation in the unknown, say, $\vec{r}$, and it is required to express $\vec{r}$ in terms of known vectors. There is no cut and dried formula to solve all such problems. Rather, the approach should be decided on the merits of the problem at hand. Often we need to take cross or dot product of the vector equation with some vector that will help isolate the unknown. Study the examples that follow carefully to learn the approach.

Ex Solve for $\vec{r}$, the vector equation $\lambda\vec{r} + \vec{a} \times \vec{r} = \vec{b}; \lambda \neq 0$.

Sol. $\lambda\vec{r} + \vec{a} \times \vec{r} = \vec{b}$ ... (i)

Taking vector product of (i) with $\vec{a}$, we have

$\lambda(\vec{r} \times \vec{a}) + (\vec{a} \times \vec{r}) \times \vec{a} = \vec{b} \times \vec{a}$

$\lambda(\vec{r} \times \vec{a}) + (\vec{a} \cdot \vec{a})\vec{r} - (\vec{r} \cdot \vec{a})\vec{a} = \vec{b} \times \vec{a}$ ...(ii)

Taking scalar product of (i) with $\vec{a}$, we have

$\lambda(\vec{r} \cdot \vec{a}) + (\vec{a} \times \vec{r}) \cdot \vec{a} = \vec{b} \cdot \vec{a}$

$\lambda(\vec{r} \cdot \vec{a}) = \vec{a} \cdot \vec{b} \Rightarrow \vec{a} \cdot \vec{r} = \frac{\vec{a} \cdot \vec{b}}{\lambda} (\lambda \neq 0)$

Using the value of $\vec{a} \cdot \vec{r}$ in (ii), we get

$\lambda(\vec{r} \times \vec{a}) + (\vec{a} \cdot \vec{a})\vec{r} - (\frac{\vec{a} \cdot \vec{b}}{\lambda})\vec{a} = \vec{b} \times \vec{a}$

$\lambda^2(\vec{r} \times \vec{a}) + \lambda(\vec{a} \cdot \vec{a})\vec{r} - (\vec{a} \cdot \vec{b})\vec{a} = -\lambda(\vec{a} \times \vec{b})$ ... (iii)

From (i), $\lambda\vec{r} - \vec{r} \times \vec{a} = \vec{b} \Rightarrow \vec{r} \times \vec{a} = \lambda\vec{r} - \vec{b}$ ... (iv)

From (iii) & (iv),

$\lambda^2(\lambda\vec{r} - \vec{b}) + \lambda(\vec{a} \cdot \vec{a})\vec{r} - (\vec{a} \cdot \vec{b})\vec{a} = -\lambda(\vec{a} \times \vec{b})$

$\lambda(\lambda^2 + a^2)\vec{r} = \lambda^2\vec{b} + (\vec{a} \cdot \vec{b})\vec{a} + \lambda(\vec{b} \times \vec{a})\{ \text{where } |\vec{a}|^2 = a^2\}$

$\vec{r} = \frac{1}{\lambda(\lambda^2 + a^2)}\{\lambda^2\vec{b} + \lambda(\vec{b} \times \vec{a}) + (\vec{a} \cdot \vec{b})\vec{a}\}$

This is the unique solution of the vector equation.