METHODS OF SOLVING ORDINARY DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DERESS

Differential Equation Of First Order And First Degree

A first-order and first-degree differential equation belongs to the type:

$$\frac{dy}{dx}+f(x,y)=0$$

This expression can also be represented as: Mdx + Ndy = 0, where M and N are functions of x and y.

Solution methods of First Order and First Degree Differential Equations

Variables Separation

Certain differential equations can be solved using the separation of variables method. This approach is applicable when the differential equation can be expressed in the form A(x) dx + B(y) dy = 0.

Where A (x) is solely a function of 'x' and B(y) is solely a function of 'y'.

A general solution for this is provided by,

$$\int A(x)\,dx + \int B(y)dy = c$$

Where 'c' is the arbitrary constant.

Ex. Find the solution to the differential equation (1 + x) y, dx = (y – 1) x , dy.

Sol. The equation can be written as –

$$\left(\frac{1+x}{x}\right)dx=\left(\frac{y-1}{y}\right)dy$$

$$\int\left(\frac{1}{x}+1\right)dx=\int\left(1-\frac{1}{y}\right)dy2$$

In x + x = y – In y + c

In y + In x = y – x + c

$$xy=e^{y-x}$$

Ex. Solve the differential equation $xy\frac{dy}{dx}=\frac{1+y^2}{1+x^2}\left(1+x+x^2\right)$.

Sol. The differential equation can be expressed as:

$$xy\frac{dy}{dx}=\left(1+y^2\right)\left(1+\frac{x}{1+x^2}\right)$$

$$\frac{y}{1+y^2}dy=\left(\frac{1}{x}+\frac{1}{1+x^2}\right)dx$$

Integrating,

$$\frac{1}{2}\ln\left(1+y^2\right)=\ln x+\tan^{-1}x+c$$

$$\sqrt{1+y^2}=cxe^{\tan^{-1}x}$$

Ex. Solve $\frac{dy}{dx}=\left(e^x+1\right)\left(1+y^2\right)$

Sol. The equation can be formulated as:

$$\frac{dy}{1+y^2}=\left(e^x+1\right)dx$$

Integrating both sides, $\tan^{-1}y = e^x + x + c$.

Ex. Solve the differential equation $\left(x^3 - y^2x^3\right)\frac{dy}{dx} + y^3 + x^2y^3 = 0$

Sol. $\left(x^3 - y^2x^3\right)\frac{dy}{dx} + y^3 + x^2y^3 = 0$

$$\frac{1-y^2}{y^3}dy + \frac{1+x^2}{x^3}dx = 0$$

$$\int\left(\frac{1}{y^3} - \frac{1}{y}\right)dy + \int\left(\frac{1}{x^3} + \frac{1}{x}\right)dx = 0$$

$$\log\left(\frac{x}{y}\right) = \frac{1}{2}\left(\frac{1}{y^2} + \frac{1}{x^2}\right) + c$$

Polar Coordinates Transformations

At times, converting to polar coordinates makes the separation of variables more convenient. In this context, it is helpful to recall the following differentials:

(A) If $x = r\cos\theta$; $y = r\sin\theta$ then,

(i) $xdx + ydy = rdr$

(ii) $dx^2 + dy^2 = dr^2 + r^2d\theta^2$

(iii) $xdy - ydx = r^2d\theta$

(B) If $x = r\sec\theta$ & $y = r\tan\theta$ then

(i) $xdx - ydy = rdr$

(ii) $xdy - ydx = r^2\sec\theta\, d\theta$.

Ex. Solve the differential equation $xdx + ydy = x(xdy - ydx)$

Sol. Taking $x = r\cos\theta$, $y = r\sin\theta$

$$x^2 + y^2 = r^2$$

$$2xdx + 2ydy = 2rdr$$

$$xdx + ydy = rdr \quad \text{......... (i)}$$

$$\frac{y}{x} = \tan\theta$$

$$\frac{x\frac{dy}{dx} - y}{x^2} = \sec^2\theta \cdot \frac{d\theta}{dx}$$

$$xdy - ydx = x^2\sec^2\theta \cdot d\theta$$

$$xdy - ydx = r^2d\theta \quad \text{........ (ii)}$$

By employing (i) and (ii) in the provided differential equation, it transforms into:

$$rdr = r\cos\theta \cdot r^2d\theta$$

$$\frac{dr}{r^2} = \cos\theta d\theta$$

$$\frac{1}{r} = \sin\theta + \lambda$$

$$-\frac{1}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} + \lambda$$

$$\frac{y+1}{\sqrt{x^2+y^2}} = c \Rightarrow (y+1)^2 = c\left(x^2+y^2\right)$$

Ex. Solve : $\dfrac{x+y\frac{dy}{dx}}{y-x\frac{dy}{dx}}=x^2+2y^2+\dfrac{y^4}{x^2}$

Sol. $\dfrac{x+y\frac{dy}{dx}}{y-x\frac{dy}{dx}}=x^2+2y^2+\dfrac{y^4}{x^2}$

$$\frac{x^4+2x^2y^2+y^4}{x^2}$$

$$\frac{xdx+ydy}{ydx-xdy}=\frac{\left(x^2+y^2\right)^2}{x^2}$$

$$\frac{xdx+ydx}{\left(x^2+y^2\right)^2}=\frac{-\left(xdy-ydx\right)}{x^2}$$

$$\frac{1}{2}\frac{2xdx+2ydy}{\left(x^2+y^2\right)^2}=-d\left(\frac{y}{x}\right)$$

$$\frac{1}{2}\frac{d\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2}=-d\left(\frac{y}{x}\right)$$

Now integrating both sides

$$-\frac{1}{2}\frac{1}{x^2+y^2}=-\frac{y}{x}+c$$

$$\frac{y}{x}-\frac{1}{2\left(x^2+y^2\right)}=c$$

Reducible to the Variables Separable form

If a differential equation can be transformed into a separable variables form through an appropriate substitution, it is termed "Reducible to the variables separable type." Its general form is. $\dfrac{dy}{dx}=f(ax+by+c)a,b\neq 0$. To solve this, put $ax + by + c = t$.

Ex. Solve $\dfrac{dy}{dx}=\cos(x+y)-\sin(x+y)$.

Sol. $\dfrac{dy}{dx}=\cos(x+y)-\sin(x+y)$

$$x + y = t$$

$$\frac{dy}{dx}=\frac{dt}{dx}-1$$

$$\frac{dt}{dx}-1=\cos t-\sin t$$

$$\int\frac{dt}{1+\cos t-\sin t}=\int dx$$

$$\int \frac{\sec^2 \frac{t}{2} dt}{2\left(1 - \tan \frac{t}{2}\right)} = \int dx$$

$$-\ln\left|1 - \tan \frac{x+y}{2}\right| = x + c$$

Ex. Solve $\frac{dy}{dy} = (4x + y + 1)^2$

Sol. Putting $4x + y + 1 = t$

$$4 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 4$$

Given equation becomes $\frac{dt}{dx} - 4 = t^2$

$$\frac{dt}{t^2 + 4} = dx$$

(Variables are separated)
Integrating both sides,

$$\int \frac{dt}{4 + t^2} = \int dx$$

$$\frac{1}{2}\tan^{-1}\frac{t}{2} = x + c$$

$$\frac{1}{2}\tan^{-1}\left(\frac{4x + y + 1}{2}\right) = x + c$$

Ex. Solve: $y' = (x + y + 1)^2$

Sol. $y' = (x + y + 1)^2$ ....(i)

Let $t = x + y + 1$

$$\frac{dt}{dx} = 1 + \frac{dy}{dx}$$

Substituting in equation (i)

We get, $\frac{dt}{dx} = t^2 + 1$

$$\int \frac{dt}{1 + t^2} = \int dx$$

$$\tan^{-1} t = x + C$$

$$t = \tan(x + C)$$

$$x + y + 1 = \tan(x + C)$$

$$y = \tan(x + C) - x - 1$$

Equation of the Form

$\Rightarrow$ $yf(xy)dx + xg(xy)dy = 0$ .... (i)

The substitution $xy = z$ transforms the differential equation of this structure into a form where the variables are separable.

Let $xy = z$ .... (ii)

$$dy = \left[\frac{xdz - zdx}{x^2}\right]$$

using equation (ii) & (iii), equation (i) becomes

$$\frac{z}{x}f(z)dx + xg(z)\left[\frac{xdz - zdx}{x^2}\right] = 0$$

$$\frac{z}{x}f(z)dx + g(z)dz - \frac{z}{x}g(z)dx = 0$$

$$\frac{z}{x}\{f(z) - g(z)\}dx + g(z)dz = 0a$$

$$\frac{1}{x}dx + \frac{g(z)dz}{z\{f(z) - g(z)\}} = 0$$

Solution by Inspection

At times, by recognizing a certain group of terms as being part of an exact differential, we can solve the differential equation in which they occur by inspection. The following list will be of help in finding a perfect differential made up of group of terms.

$$d(xy) = xdy + ydx$$

$$d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$$

$$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$

$$d\left(\ln\frac{y}{x}\right) = \frac{xdy - ydx}{xy}$$

$$d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$$

$$d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx - xdy}{x^2 + y^2}$$

$$d\left(\frac{y^2}{x}\right) = \frac{2xydy - y^2dx}{x^2}$$

$$d\left(\frac{x^2}{y}\right) = \frac{2xydx - x^2dy}{y^2}$$

$$d\left(\frac{1}{xy^2}\right) = -\frac{y^2dx + 2xydy}{x^2y^4}$$

$$d\{\sin^{-1}(xy)\} = \frac{xdy + ydx}{\sqrt{1 - x^2y^2}}$$

Homogeneous Equations

A function f(x, y) is considered a homogeneous function of degree n if the substitution $x = \lambda x$, $y = \lambda y$, , $\lambda > 0$ results in the equality:

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

The degree of homogeneity, denoted by 'n', can take any real number.

Ex. Find the degree of homogeneity for the given function.

(i) $f(x, y) = x^2 + y^2$ (ii) $f(x, y) = \left(\dfrac{x^{\frac{3}{2}} + y^{\frac{3}{2}}}{(x+y)}\right)$ (iii) $f(x, y) = \sin\left(\dfrac{x}{y}\right)$

Sol. (i)

$$f(\lambda x, \lambda y) = \lambda^2 x^2 + \lambda^2 y^2$$
$$= \lambda^2\left(x^2 + y^2\right)$$
$$= \lambda^2 f(x, y)$$

Degree of homogeneity $\rightarrow 2$

(ii)

$$f(\lambda x, \lambda y) = \frac{\lambda^{3/2}x^{3/2} + \lambda^{3/2}y^{3/2}}{\lambda x + \lambda y}$$
$$f(\lambda x, \lambda y) = \lambda^{1/2} f(x, y)$$

Degree of homogeneity $\rightarrow$ ½

(iii)

$$f(\lambda x, \lambda y) = \sin\left(\frac{\lambda x}{\lambda y}\right)$$
$$= \lambda^{\circ}\sin\left(\frac{x}{y}\right)$$
$$= \lambda^{\circ} f(x,y)$$

Degree of homogeneity $\rightarrow 0$

Ex. Ascertain whether each of the following functions is homogeneous or not.

(i) $f(x, y) = x^2 - xy$ (ii) $f(x, y) = \dfrac{xy}{x + y^2}$ (iii) $f(x, y) = \sin xy$

Sol. (i) $= \lambda^2 x^2 - \lambda^2 xy$

$= \lambda^2\left(x^2 - xy\right) = \lambda^2 f(x, y)$ Homogeneous.

(ii) $f(\lambda x, \lambda y) = \dfrac{\lambda^2 xy}{\lambda x + \lambda^2 y^2} \neq \lambda^n f(x, y)$ Not homogeneous.

(i) $f(\lambda x, \lambda y) = \sin\left(\lambda^2 xy\right) \neq \lambda^{n_f}(x, y)$ Not homogeneous.

Homogeneous first order differential equation

A differential equation of the form $\dfrac{dy}{dx} = \dfrac{f(x, y)}{g(x, y)}$

Where f (x,y) and g(x,y) are homogeneous functions of x,y and of the same degree, is said to be homogeneous. Such equations can be solved by substituting $y = vx$, A function F(x, y) where f(x, y) and g (x, y) are homogeneous functions of x and y and of the same degree, is considered homogeneous. Such equations can be solved by substituting y = vx.

(i) So that the dependent variable y is changed to another variable v.

As f(x, y) and g(x, y) are homogeneous functions of the same degree, denoted as n, they can be expressed as:

$$f(x,y) = x^n f_1\left(\frac{y}{x}\right)$$

$$g(x,y) = x^n g_1\left(\frac{y}{x}\right)$$

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

The given differential equation, therefore, becomes

$$v + x$$

$$v + x\frac{dv}{dx} = \frac{f_1(v)}{g_1(v)}$$

$$\frac{g_1(v)dv}{f_1(v) - vg_1(v)} = \frac{dx}{x},$$

So that the variables v and x are now separable.

In certain cases, homogeneous equations can be resolved by substituting (x = vy) or by employing polar coordinate substitution.

Ex. Find the solution for the differential equation $(x^2 - y^2)\,dx + 2xydy = 0$, with the initial condition y = 1 when x = 1.

Sol.

$$\frac{dy}{dx} = -\frac{x^2 - y^2}{2xy}$$

$$y = ux$$

$$\frac{dy}{dx} = v + \frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = -\frac{1 - v^2}{2v}$$

$$\int \frac{2v}{1+v^2}dv = -\int \frac{dx}{x}$$

$$\text{In}\left(1+v^2\right) = -\text{In}x + c$$

$$x=1,\ y=1,\ \ v=1$$

$$\text{In } 2 = c$$

$$\text{In}\left\{\left(1+\frac{y^2}{x^2}\right).x\right\} = \text{In}2$$

$$x^2 + y^2 = 2x$$

Ex. Find the solution to the differential equation: $\left(1+2e^{\frac{x}{y}}\right)dx+2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$

Sol. The equation is homogeneous with a degree of 0.

Put $x = vy, \quad dx = vdy + ydv$

Subsequently, the differential equation transforms into

$$\left(1+2e^{v}\right)(vdy+ydv)+2e^{v}(1-v)dy=0$$

$$\left(v+2e^{v}\right)dy+y\left(1+2e^{v}\right)dv=0$$

$$\frac{dy}{y}+\frac{1+2e^{v}}{v+2e^{v}}dv=0$$

Integrating and replacing v by $\frac{x}{y}$,

We get $\ln y+\ln\left(v+2e^{V}\right)=\ln c$

And $x+2ye^{\frac{x}{y}}=c$

Non-homogeneous Equation of First Degree in X and Y

Equations reducible to homogeneous form

An equation of the form where $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$ can be reduced to homogeneous form by changing the variables x, y to u, v as $x = u + h, y = v + k$

Where h, k are constants chosen to make the given equation homogeneous.

We have $\frac{dy}{dx}=\frac{dv}{du}$

The equation becomes,

$$\frac{dv}{du}=\frac{a_1u+b_1v+\left(a_1h+b_1k+c_1\right)}{a_2u+b_2v+\left(a_2h+b_2k+c_2\right)}$$

Let, h and k be chosen so as to satisfy the equation

$$a_1h+b_1k+c_1=0 \quad \text{... (i)}$$

$$a_2h+b_2k+c_2=0 \quad \text{... (ii)}$$

Solve for h and k from (i) and (ii)

Now, $\frac{du}{dv}=\frac{a_1u+b_1v}{a_2u+b_2v}$

Is a homogeneous equation and can be solved by substituting v = ut.

Ex. Solve the differential equation $\frac{dy}{dx}=\frac{x+2y-5}{2x+y-4}$

Sol. Let $x = x + h, y = Y + k$

$$\frac{dy}{dX}=\frac{d}{dX}(Y+k)$$

$$\frac{dy}{dX}=\frac{dY}{dX}$$

$$\frac{dx}{dX}=1+0$$

On dividing (i) by (ii)

$$\frac{dy}{dx}=\frac{dY}{dx}$$

$$\frac{dY}{dX}=\frac{X+h+2\left(Y+k\right)-5}{2X+2h+Y+k-4}$$

$$=\frac{X+2Y+\left(h+2k-5\right)}{2X+Y+\left(2h+k-4\right)}$$

h & k are such that

$$h + 2k - 5 = 0$$
$$2h + k - 4 = 0$$
$$h = 1, k = 2$$

$\frac{dY}{dX}=\frac{X+2Y}{2X+Y}$ Which is homogeneous differential equation.

Now, substituting $Y = vX$

$$\frac{dY}{dX}=v+X\frac{dv}{dX}$$

$$X\frac{dv}{dX}=\frac{1+2v}{2+v}-v$$

$$\int\frac{2+v}{1-v^2}dv=\int\frac{dX}{X}$$

$$\int\left(\frac{1}{2\left(v+1\right)}+\frac{3}{2\left(1-v\right)}\right)dv=\text{In}X+c$$

$$\frac{1}{2}\ln(v+1)-\frac{3}{2}\ln(1-v)=\ln X+c$$

$$\ln\left|\frac{v+1}{(1-v)^3}\right|=\ln X^2+2c$$

$$\frac{(Y+X)}{(X-Y)^3}\frac{X^2}{X^2}=e^{2c}$$

$$X+Y=c'(X-Y)^3$$

$$e^{2c}=c^1$$

$$x-1+y-2=c'(x-1-y+2)^3$$

$$x+y-3=c'(x-y+1)^3$$

If $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$ then the substitution $ax + by = v$ will reduce it to the form in which variables are separable.

Ex. Solve: $(x + y)dx + (3x + 3y - 4)\,dy = 0$

Sol. Let

$$t = x + y$$
$$dy = dt - dx$$
$$tdx + (3t - 4)(dt - dx) = 0$$

$$2dx+\left(\frac{3t-4}{2-t}\right)dt=0$$

$$2dx-3dt+\frac{2}{2-t}dt=0$$

Integrating and replacing t by x + y, we get

$$2x - 3t - 2[\ln|(2 - t)|] = c_1$$

$$2x - 3(x + y) - 2[\ln|(2 - x - y)|] = c_1$$

$$x + 3y + 2\ln|(2 - x - y)| = c$$

Ex. Evaluate $\frac{dy}{dx}=\frac{2x+3y-1}{4x+6y-5}$

Sol. Putting $u = 2x + 3y$

$$\frac{1}{3}\left(\frac{du}{dx}-2\right)=\frac{u-1}{2u-5}$$

$$\frac{du}{dx}=\frac{3u-3+4u-10}{2u-5}$$

$$\int\frac{2u-5}{7u-13}dx=\int dx$$

$$\frac{2}{7}\int 1\cdot du-\frac{9}{7}\int\frac{1}{7u-13}\cdot du=x+c$$

$$\frac{2}{7}u-\frac{9}{7}\cdot\frac{1}{7}\ln(7u-13)=x+c$$

$$4x+6y-\frac{9}{7}\ln(14x+21y-13)=7x+7c$$

$$-3x+6y-\frac{9}{7}\ln(14x+21y-13)=c'$$

If $a_2 + b_1 = 0$, then a simple cross multiplication and substituting d (xy) for xdy + ydx and integrating term by term, yield the results easily.

Ex. Evaluate $\frac{dy}{dx}=\frac{x-2y+1}{2x+2y+3}$

Sol.

$$\frac{dy}{dx}=\frac{x-2y+1}{2x+2y+3}$$

$$2xdy+2ydy+3dy=xdx-2ydx+dx$$

$$(2y+3)dy=(x+1)dx-2(xdy+ydx)$$

On integrating,

$$\int(2y+3)dy=\int(x+1)dx-\int 2\,d(xy)$$

$$y^2+3y=\frac{x^2}{2}+x-2xy+c$$

Ex. Find $\frac{dy}{dx}=\frac{x-2y+5}{2x+y-1}$

Sol. Cross multiplying,

$$2xdy+ydy-dy=xdx-2ydx+5dx$$

$$2(xdy+ydx)+ydy-dy=xdx+5dx$$

$$2d(xy)+ydy-dy=xdx+5dx$$

On integrating,

$$2xy + \frac{y^2}{2} - y = \frac{x^2}{2} + 5x + c$$

$$x^2 - 4xy - y^2 + 10x + 2y = c'$$

$$c' = -2c$$

Ex. Solve $y(xy+1)dx + x\left(1 + xy + x^2y^2\right)dy = 0a$

Sol. Let $xy = v$

$$y = \frac{v}{x}$$

$$dy = \frac{xdv - vdx}{x^2}$$

Now, differential equation becomes

$$\frac{v}{x}(v+1)dx + x\left(1 + v + v^2\right)\left(\frac{xdv - vdx}{x^2}\right) = 0$$

On solving, we get

$$v^3dx - x\,(1 + v + v^2)\,dv = 0$$

Separating the variables & integrating

We get, $$\int\frac{dx}{x} - \int\left(\frac{1}{v^3} + \frac{1}{v^2} + \frac{1}{v}\right)dv = 0$$

$$\ln x + \frac{1}{2v^2} + \frac{1}{v} - \ln v = c$$

$$2v^2 \ln\left(\frac{v}{x}\right) - 2v - 1 = -2cv^2$$

$$2x^2y^2 \ln n - 2xy - 1 = Kxx^2y^2$$

where $K = -2c$

Linear Differential Equation (Lagrange's Linear Differential Equation)

A linear differential equation has the following characteristics:

1. The dependent variable and its derivative are of the first degree and not multiplied together.
2. All derivatives should be in polynomial form.
3. The order of the derivatives may be more than one.

The m^{th} order linear differential equation is of the form.

$$P_0(x)\frac{d^m y}{dx^m}+P_1(x)\frac{d^{m-1}y}{dx^{min-1}}+\ldots\ldots\ldots\ldots\ldots+P_{m-1}(x)\frac{dy}{dx}+P_m(x)y=\phi(x)$$

Where $P_0(x)$, $P_1(x)$ ....$P_m(x)$ are called the coefficients of the differential equation.

The coefficients of the differential equation are denoted as $P_0(x)$, $P_1(x)$ ....$P_m(x)$.

Note: While a linear differential equation is always of the first degree, it's important to note that not every differential equation of the first degree is linear.

E.g. the differential equation $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+y^2=$ is not linear, though its degree is 1.

$\frac{dy}{dx}+y^2\sin x=\ln x$ is not a Linear differential equation.

Ex. Which of the following equations is linear and which one is nonlinear?

(A) $\frac{dy}{dx}+xy^2=1$ (B) $x^2\frac{dy}{dx}+y=e^x$ (C) $\frac{dy}{dx}+3y=xy^2$

(D) $x\frac{dy}{dx}+y^2=\sin x$ (E) $\frac{dy}{dx}=\cos x$ (F) $\frac{d^2y}{dx^2}+y=0$

(G) $dx+dy=0$ (H) $x\left(\frac{dy}{dx}\right)+\frac{3}{\left(\frac{dy}{dx}\right)}=y^2$

Sol. None Linear (A), (C), (D) , (H)

Linear (B), (E), (F),(G)

Linear differential equations of first order

The differential equation $\frac{dy}{dx}+Py=Q$ is linear in y.

(Where P and Q are functions solely dependent on x.)

Integrating Factor (I.F.) :

It is an expression that, when multiplied by a differential equation, transforms it into an exact form.

The integrating factor for a linear differential equation is $=g(x)=e^{\int Pdx}$ After multiplying the above equation by (ignoring the constant of integration),

I.F it becomes;

$$\frac{dy}{dx}\cdot e^{\int Pdx}+Py\cdot e^{\int Pdx}=Q\cdot e^{\int Pdx}$$

$$\frac{d}{dx}\left(y\cdot e^{\int Pdx}\right)=Q\cdot e^{\int Pdx}$$

$$e^{\int Pdx}=\int Q\cdot e^{\int Pdx}+C$$

At times, the differential equation becomes linear when x is considered the dependent variable and y as the independent variable. In such cases, the differential equation takes the following form:

$$\frac{dx}{dy} + P_1 x = Q_1$$

Here, P_1 and Q_1 are functions of y. The I.F. now is $e^{\int P_1 dy}$

Ex. Find $\left(1+y^2\right)+\left(x-e^{tn^{-1}y}\right)\frac{dy}{dx}=0$

Sol. The differential equation can be expressed as:

$$\left(1+y^2\right)\frac{dx}{dy}+x=e^{\tan^{-1}y}$$

$$\frac{dx}{dy}+\frac{1}{1+y^2}\cdot x=\frac{e^{\tan^{-1}y}}{1+y^2}$$

So, solution is $xe^{\tan^{-1}y}=\int\frac{e^{\tan-1}ye^{\tan^{-1}y}}{1+y^2}dy$

Let, $e^{\tan-1}y=t$

$$\frac{e^{\tan-1}y}{1+y^2}dy=dt$$

$$xe^{\tan^{-1}y}=\int tdt$$

Putting $e^{\tan-1}y=t$

Or $xe^{\tan-1}y=\frac{t^2}{2}+\frac{c}{2}$

$$2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+c$$

Ex. Solve $\frac{dy}{dx}+\frac{3x^2}{1+x^3}y=\frac{\sin^2 x}{1+x^3}$

Sol. $\frac{dy}{dx}+Py=Q$

$$P=\frac{3x^2}{1+x^3}$$

$$F=e^{\int P\cdot dx}=e^{\int\frac{3x^2}{1+x^3}dx}=e^{\ln\left(1+x^3\right)}=1+x^3$$

General solution is

$$y(\,IF)=\int Q(\,IF)\cdot dx+c$$

$$y\left(1+x^3\right)=\int\frac{\sin^2 x}{1+x^3}\left(1+x^3\right)dx+c$$

$$y\left(1+x^3\right)=\int\frac{1-\cos 2x}{2}dx+c$$

$$y\left(1+x^3\right)=\frac{1}{2}x-\frac{\sin 2x}{4}+c$$

Ex. Evaluate $x\ln x\frac{dy}{dx}+y=2\ln x$

Sol. $\frac{dy}{dx}+y=\frac{2}{x}$

$$P = \frac{1}{x/\ln x}, Q = \frac{2}{x}$$

$$IF = e^{\int P.dx} = e^{\int \frac{1}{x \ln x} dx} = e^{\ln(\ln x)} = \ln x$$

General solution is

$$y \cdot (\ln x) = \int \frac{2}{y} \ln x \cdot dx + c$$

$$y(\ln x) = (\ln x)^2 + c$$

Equations reducible to linear form

By change of variable.

Frequently, a differential equation can be transformed into a linear form through a suitable substitution of the non-linear term.

Ex. Solve: $\frac{dy}{dx} = \cos x\left(\sin x - y^2\right)$

Sol. The provided differential equation can be transformed into a linear form through a change of variable by an appropriate substitution.

Substituting $y^2 = z$

$$2y\frac{dy}{dx} = \frac{dz}{dx}$$

differential equation becomes

$$\frac{\sin x}{2}\frac{dz}{dx} + \cos x \cdot z = \sin x \cos x$$

$\frac{dz}{dx} + 2\cot x \cdot z = 2\cos x$ which is linear in $\frac{dz}{dx}$

$$IF = e^{\int 2\cos x dx} = e^{2 \min x} = \sin^2 x$$

General solution is

$$z.\sin^2 x = \int 2\cos x \cdot \sin^2 x \cdot dx + c$$

$$y^2 \sin^2 x = \frac{2}{3}\sin^3 x + c$$

Bernoulli's equation

Equations of the form $\frac{dy}{dx} + Py = Q \cdot y^n, n \neq 0$ and $n \neq 1$ where P and Q are functions of x, is called Bernoulli's equation.

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e.g. $2\sin x \frac{dy}{dy} - y\cos x = xy^3 e^x$

On dividing by y^n

We get $y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q$

Let $y^{n-1} = t,$

So that $(-n+1)y^{-1}\frac{dy}{dx} = \frac{dt}{dx}$

Then equation becomes $\frac{dt}{dx} + P(1-n)t = Q(1-n)$ linear with t as a dependent variable.

Ex. Find $\frac{dy}{dx}-\frac{y}{x}=\frac{y^2}{x^2}$ (Bernoulli's equation)

Sol. Dividing both sides by y^2

$$\frac{1}{y^2}\frac{dy}{dx}-\frac{1}{xy}=\frac{1}{x^2}2$$

Putting $\frac{1}{y}=t$

$$-\frac{1}{y^2}\frac{dy}{dx}=\frac{dt}{dx}$$

Differential equation (i) becomes,

$$-\frac{dt}{dx}-\frac{t}{x}=\frac{1}{x^2}$$

$\frac{dt}{dx}+\frac{t}{x}=-\frac{1}{x^2}$ which is linear differential equation in $\frac{dt}{dx}$

$$=e^{\int\frac{1}{x}dx}=e^{\ln x}=x$$

$$t\cdot x=\int-\frac{1}{x^2}\cdot x\,dx+c$$

$$tx=-\ln x+c$$

$$\frac{x}{y}=-\ln x+c$$

Ex. Determine the solution for the given differential equation: $\frac{dy}{dx}-y\tan x=-y^2\sec x$

Sol. $\frac{1}{y^2}\frac{dy}{dx}-\frac{1}{y}\tan x=-\sec x$

$$\frac{1}{y}=v;\frac{-1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$$

$$\frac{-dv}{dx}-v\tan x=-\sec x$$

$$\frac{dv}{dx}+v\tan x=\sec x,$$

Here, $P=\tan x, Q=\sec x$

I.F. $=e^{\int \text{sanxdx}}=|\sec x|\ v\,|\sec x|=\int\sec^2 x\,dx+c$

Hence the solution is $y^{-1}|\sec x|=\tan x+c$